Lecture 36 Applications of the Poisson integral, Riemann removable sing. for harmonic fcns

Exam 2 due in Gradescope tonight by 11:00 pm

Thm Suppose u is a continuous real valued fcn on a domain Ω that satisfies the averaging property: $u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + Re^{it})\,dt$ when $\overline{D_R(z_0)} \subset \Omega$. Then u is harmonic on Ω.

Pf Being harmonic is a local property. So can restrict attention to a disc in Ω, $\overline{D_R(z_0)} \subset \Omega$. Can change vars: $f(z) = \frac{z-z_0}{R}$.

Can reduce our attention to $\overline{D_1(0)}$.

Classic trick Let $\tilde{u}(z) = \int_0^{2\pi} P(z,\theta)\, u(e^{i\theta})\, d\theta$. Know $\tilde{u}$ is continuous on $\overline{D_1(0)}$ and harmonic in $D_1(0)$ and $u(e^{i\theta}) = \tilde{u}(e^{i\theta})$ on $C_1(0)$.

Aha! u satisfies ave prop too. $\tilde{u} - u$ satisfies ave prop.

[Idea: Ave prop $\Rightarrow$ max princ. See $\tilde{u} - u \le 0$ on $\overline{D_1(0)}$.
Repeat with $u - \tilde{u}$ in place of $\tilde{u} - u$. Get $u \equiv \tilde{u}$ $\leftarrow$ harmonic!]

Details: $\tilde{u} - u$ is continuous on $\overline{D_1(0)}$, so it assumes its max M.

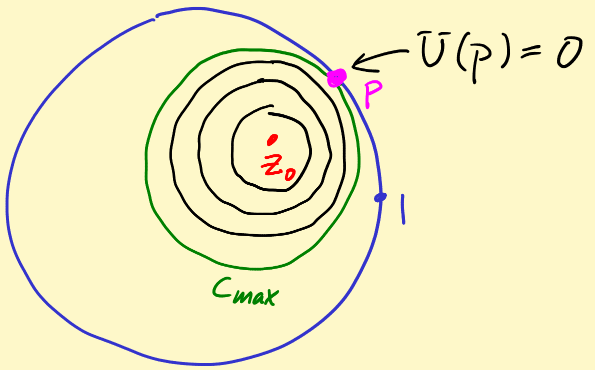
Want to see $M \le 0$. Assume $M > 0$. [Know $\tilde{u} - u = 0$ on $C_1(0)$.]

Say $U(z_0) = \tilde{u}(z_0) - u(z_0) = M$.

Note: $z_0 \notin C_1(0)$.

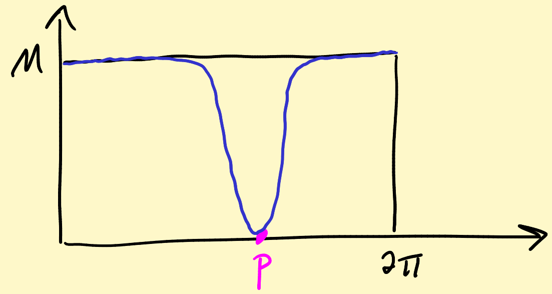
Ave prop holds on biggest circle C_{max}:

(Use uniform continuity and limits as circles expand.)



$U \le M$ on C_{max}, $U(p) = 0$.

$$M = U(z_0) = \left[\text{ave } U \text{ on } C_{max}\right] \le M$$



Since $U(p)=0$, ave must be $< M$. $\lightning$.

So M can't be >0.

Conclude $\tilde{u} - u \le 0 = \text{Max}\{\tilde{u} - u \text{ on } C_1(0)\}$.

Repeat using $u - \tilde{u}$ in place $\tilde{u} - u$. Get $\tilde{u} - u \equiv 0$ on $\overline{D_1(0)}$.

So $u = \tilde{u}$, harmonic. ✓

Riemann removable singularity thm for harmonic fcns.

Assume u is a real valued harmonic fcn on $D_R(z_0) - \{z_0\}$ that is bounded ($|u| < M$ on set). Then z_0 is a "removable singularity", meaning we can give u a value at z_0 to make it harmonic on $D_R(z_0)$.

EX $\operatorname{Ln}|z|$ is harmonic on $D_1(0) - \{0\}$, no harmonic conjugate.

$\operatorname{Ln}|z| \le 0$ on $\overline{D_1(0)} - \{0\}$ and $\lim\limits_{|z| \to 0} \operatorname{Ln}|z| = -\infty$.

Pf: Can assume u is harmonic on $D_r(0) - \{0\}$ with $r > 1$. Say $|u| < M$ on that set. Let $\tilde{u}$ = Poisson integral of $u(e^{i\theta})$ on $C_1(0)$. $\tilde{u} = u$ on $C_1(0)$.

Fiendishly clever sneaky trick: Define

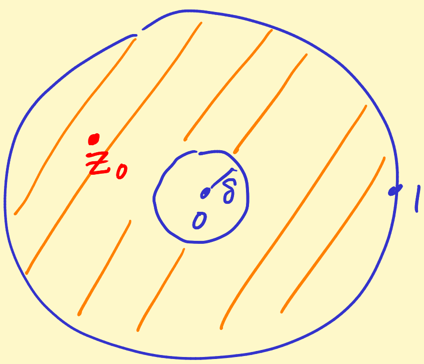
$$v = \underbrace{u - \tilde{u}}_{\substack{\text{bounded} \\ =0 \text{ on } C_1(0)}} + \underbrace{\varepsilon \operatorname{Ln}|z|}_{\substack{\text{harmonic on } D_1(0)-\{0\} \\ =0 \text{ on } C_1(0)}} \qquad (\varepsilon > 0)$$

Aha! v is harmonic on $D_1(0) - \{0\}$,

$v \equiv 0$ on $C_1(0)$,

$$\lim_{|z|\to 0} v(z) = -\infty$$

Feeling: Must be a max princ for this situation. There is!



$v(z) \to -\infty$ as $|z| \to 0$.
So $\exists\ \delta$ with $0 < \delta < 1$
such that $v(z) < -1$
when $|z| < \delta$.

Pick z_0 in $A_\delta^1(0)$. Apply max princ#2 on $A_\delta^1(0)$.

Max attained on boundary: $\quad v(z_0) \le \operatorname{Max}\{0, -1\} = 0$

<u>Note</u> $v < -1$ on $D_\delta(0) - \{0\}$.

So $v \le 0$ on $\overline{D_1(0)} - \{0\}$.

Hmmm. True for <u>any</u> $\varepsilon > 0$! Can let $\varepsilon \searrow 0$.

$$v(z) = u(z) - \tilde{u}(z) + \underbrace{\varepsilon \operatorname{Ln}|z|}_{\to 0 \text{ as } \varepsilon \searrow 0} \le 0$$

So $\quad u(z) - \tilde{u}(z) \le 0$ on $\overline{D_1(0)} - \{0\}$.

Repeat using $\tilde{u}-u$ in place of $u-\tilde{u}$. See $u=\tilde{u}$ on punctured disc. If we set $u(0)=\frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\,d\theta$, u is harmonic on $D_1(0)$.

Fun fact Solution to Dirichlet problem on $D_1(0)$ with polynomial boundary data is a harmonic polynomial.

Pf
$$\underbrace{p(x,y)}_{\text{real valued}} = p\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) = \sum_{n,m=0}^{N} c_{nm} z^n \bar{z}^m$$
← looks complex valued, but imag part $\equiv 0$.

$$z^n \bar{z}^m = \begin{cases} |z|^{2n} = 1 & \text{on } C_1(0) \text{ if } n=m \leftarrow \text{extend by } 1 \\ z^{n-m} \underbrace{\bar{z}^m z^m}_{=1 \text{ on } C_1(0)} = z^{n-m} & \text{on } C_1(0) \text{ if } n>m \leftarrow \text{extend } z^{n-m} \\ \underbrace{z^n \bar{z}^n}_{=1 \text{ on } C_1(0)} \bar{z}^{m-n} = \bar{z}^{m-n} & \text{on } C_1(0) \text{ if } m>n \leftarrow \text{extend } \bar{z}^{m-n} \end{cases}$$

Get extension $q_1(z) + q_2(\bar{z})$, q_1, q_2 polys.

$$\text{Sol}^n \text{ to D prob} = \underbrace{\text{Real part}\left[q_1(z)+q_2(\bar{z})\right]}_{\text{real poly in } x,y\,!}$$

Ellipses have polynomial ⇔ property for D. Prob.

Khavinson-Schapiro conjecture: Only discs and ellipses have poly ⇔ property. Open problem!