Lecture 37 Weak averaging property, harmonic Schwarz reflection

Polynomial data on $C_1(0)$: $\underset{\uparrow \text{ real poly}}{p(x,y)} = p\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) = \sum c_{nm} z^n \bar{z}^m$ $= u + iv$, $v \equiv 0$, $c_{nm} \in \mathbb{C}$

$$z^n \bar{z}^m = \begin{cases} 1 & \text{on } C_1(0) \text{ when } n=m \\ z^{n-m} & \text{on } C_1(0) \text{ when } n>m \\ \bar{z}^{m-n} & \text{on } C_1(0) \text{ when } m>n \end{cases}$$

real & imag parts are harmonic on $D_1(0)$

Use these to get harmonic extension $q_1(z) + q_2(\bar{z})$, q_1, q_2 polys

$$p(x,y) = \underbrace{\mathrm{Re}\left[q_1(x+iy) + q_2(x-iy)\right]}_{\text{harmonic poly in } x,y \text{ solves D. prob!}} \quad \text{on } C_1(0)$$

[Remark: $\mathrm{Im}\left[q_1(z) + q_2(\bar{z})\right] \equiv 0$ by max princ.]

Khavinson-Shapiro Conjecture Discs and ellipses are the only domains with (poly data) $\longmapsto$ (poly sol^{n}s to D. prob)

Fact Discs also have (rational data) $\longmapsto$ (rational sol^{n}s) property.

Thm (B, Kh, Sh, Ebenfelt) Only discs have this property!

Thm Suppose u_k are real valued harmonic fcns on a domain Ω and $u_k \rightrightarrows u$. Then u is harmonic.

Furthermore $\dfrac{\partial^{n+m}}{\partial x^n \partial y^m} u_K \rightrightarrows \dfrac{\partial^{n+m}}{\partial x^n \partial y^m} u$ on Ω

Pf MA 504: unif limits of continuous fcns are continuous.

so u cont on Ω. Easy: Ave prop for $u_K \Rightarrow$

u has ave prop $\Rightarrow$ u is harmonic!

$$\frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\,d\theta = \lim_{K\to\infty} \int \cdots u_K \cdots = \lim u_K(z_0) = u(z_0) \checkmark$$

unif conv on $C_r(z_0)$, compact

To see derivatives $\rightrightarrows$, differentiate under $\int$ in Poisson formula, cover by "shrunken" discs, get finite subcover, etc.

Remark $P(z, e^{i\theta}) = \dfrac{1}{2\pi} \dfrac{1-|z|^2}{|z-e^{i\theta}|^2} = \operatorname{Re}\left[\dfrac{1}{2\pi} \dfrac{z+e^{i\theta}}{z-e^{i\theta}}\right]$

$$u = \mathcal{P}u = \operatorname{Re}\left[\frac{1}{2\pi}\int_0^{2\pi} \frac{z+e^{i\theta}}{z-e^{i\theta}}\, u(e^{i\theta}) \frac{ie^{i\theta}}{ie^{i\theta}}\, d\theta\right]$$

$$= \operatorname{Re}\Bigg[\underbrace{\frac{1}{2\pi}\int_{C_1(0)} \frac{z+w}{z-w}\, u(w) \frac{1}{iw}\, dw}_{\text{analytic in } z}\Bigg]$$

HWK 2: prob 1.

Remark Take Im part, see $u \mapsto v$ = harm conj of u!

"Hilbert transform" very interesting. u cont., v not bndd possible. But $u \in L^2(C_1(0)) \Rightarrow v \in L^2$: Bounded operator!

<u>Weak averaging property</u> A cont fcn u on a domain Ω has the WAProp if, for each pt $a \in \Omega$, there <u>exists</u> $r > 0$ such that $D_r(a) \subset \Omega$ and

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a + \rho e^{it})\,dt \quad \text{for } 0 < \rho < r.$$

[Note: r depends on a, and can be smaller than $\operatorname{dist}(a, b\Omega)$.]

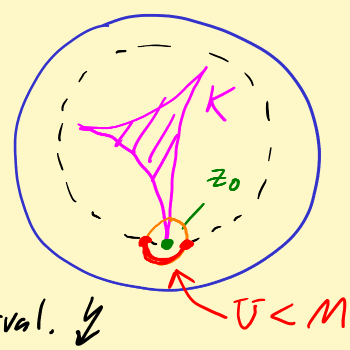
<u>Thm</u> WAProp $\Rightarrow$ harmonic

<u>Pf</u> This is a "local" thm, so we reduce to u continuous on $D_R(0)$, $R > 1$, u satisfies WAProp there.

Let $\tilde{u}(z) = \int_0^{2\pi} P(z, e^{i\theta})\,u(e^{i\theta})\,d\theta$. Know $\tilde{u} = u$ on $C_1(0)$, $\tilde{u}$ harmonic in $D_1(0)$. Let $U = \tilde{u} - u$ and

$M = \underset{\overline{D_1(0)}}{\operatorname{Max}}\, U$. Suppose $M > 0$. Let $K = \{z \in \overline{D_1(0)} : U(z) = M\}$

Know K compact. Know $C_1(0) \cap K = \emptyset$. So K is compact subset of $D_1(0)$. $|z|$ is a continuous fcn on K, so it attains its max r_m at a pt $z_0 \in K$.



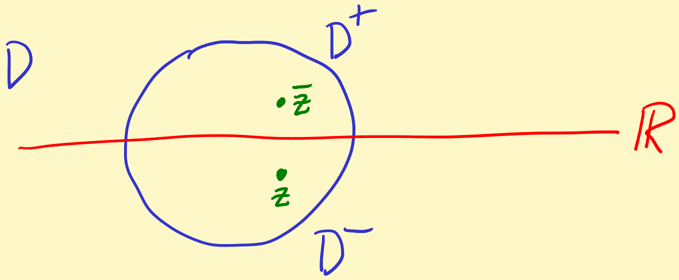
Have ave prop on small circle $\longrightarrow$

$M = u(z_0) =$ Ave of stuff $\leq M$ and $< M$ on open interval. ↯

So $M=0$. Repeat using $u-\tilde{u}$ in place of $\tilde{u}-u$.

Get $u=\tilde{u}$ ← harmonic on $D_1(0)$ ✓

Schwarz reflection principle for harmonic fcns Suppose u is a real valued harmonic fcn on D^+ that is continuous up to $\mathbb{R}\cap D$ and vanishes on $\mathbb{R}\cap D$.

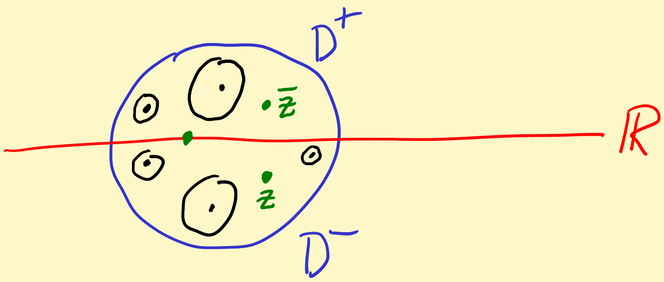


Then
$$U(z)=\begin{cases} u(z) & \text{on } D^+ \\ 0 & \text{on } \mathbb{R}\cap D \\ -u(\bar{z}) & \text{on } D^- \end{cases}$$

is harmonic on D.

$$\Delta\left(-u(x,-y)\right)\equiv 0$$
by chain rule

Pf



weak ave prop holds at green dot on $\mathbb{R}$ too!