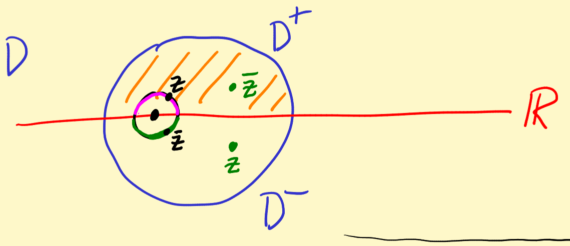
# Lecture 38 The general Cauchy theorem and integral formula

Schwarz reflection principle for harmonic fcns Suppose $u$ is a real valued harmonic fcn on $D^+$ that is continuous up to $\mathbb{R} \cap D$ and vanishes on $\mathbb{R} \cap D$.




Then
$$U(z) = \begin{cases} u(z) & \text{on } D^+ \\ 0 & \text{on } \mathbb{R} \cap D \\ -u(\bar{z}) & \text{on } D^- \end{cases}$$
is harmonic on $D$.

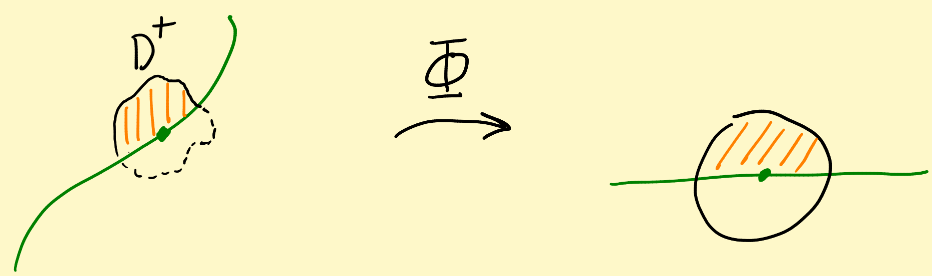
Pf $\Delta\left[-u(x,-y)\right] = -(\Delta u)(x,-y) \equiv 0$

$u=0$ on $\mathbb{R}$, cont up to $\mathbb{R} \Rightarrow U$ cont on $D$.

Ave prop at $x_0 \in \mathbb{R}$: Top half $\int$ cancels bottom half. ✓

Weak ave prop $\Rightarrow U$ harmonic

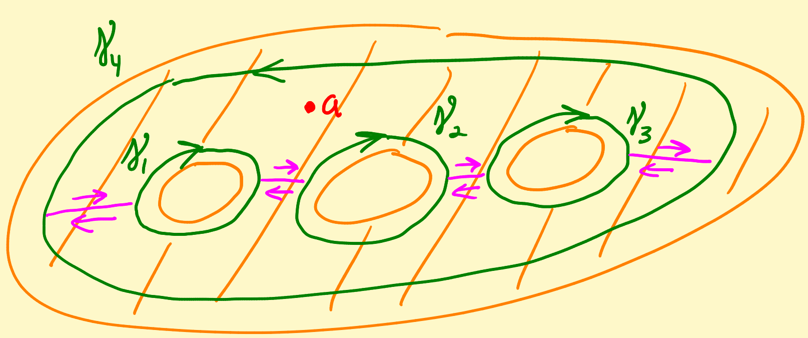
Look at extension of analytic fcn Schwarz reflection. Get




$\gamma$ real analytic curve

$u$ harmonic on $D^+$, cont up to $\gamma$, $0$ on $\gamma$. Then $u$ extends harmonically past $\gamma$

## General Cauchy theorem




$f$ analytic

Feeling $\Gamma = \bigcup_{k=1}^{4} \gamma_k$ is a "chain". ["Cycle" when $\gamma$'s are closed]

Write $\int_\Gamma f\,dz = \sum_{k=1}^{4} \int_{\gamma_k} f\,dz$

$$\mathrm{Ind}_\Gamma(a) = \frac{1}{2\pi i}\int_\Gamma \frac{1}{z-a}\,dz = \sum_{k=1}^{4} \mathrm{Ind}_{\gamma_k}(a)$$

Standard Cauchy thm: $\int_\Gamma f\,dz = 0.$

[Top half + Bottom half: Slits cancel, leaving $\Gamma$.]

Exciting: Apply to $\frac{f(z)}{z-a}$ instead of $f$.

Top half: $= f(a)$
Bottom half: $= 0$

Cauchy integral formula: $f(a) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-a}\,dz$

<u>Fact</u> $\mathrm{Ind}_\Gamma(a)$ = constant on connected components of $\mathbb{C} - \mathrm{tr}(\Gamma)$. In fact, it is an integer.

<u>Why</u>: $\frac{d}{da}\mathrm{Ind}_\Gamma(a) = \underbrace{\frac{1}{2\pi i}\int_\Gamma \frac{1}{(z-a)^2}\,dz}_{=0}$ So $\mathrm{Ind}_\Gamma$ is constant on components.

Argument principle consequence using $f(z) = z-a$ shows $\mathrm{Ind}_{\gamma_k}(a)$ is an integer.

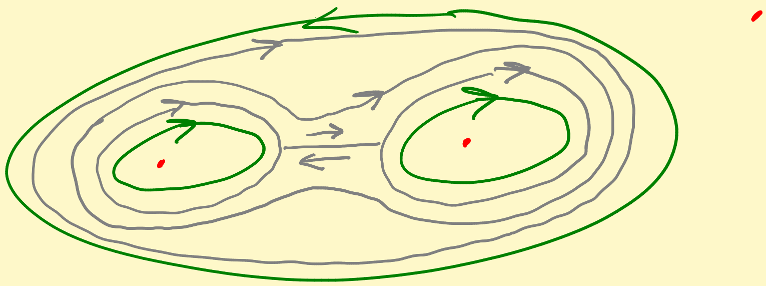
<u>Big hypothesis for general Cauchy thm</u> $\Gamma$ is a cycle (meaning $\Gamma = \bigcup_{k=1}^{N} \gamma_k$ where $\gamma_k$ are closed curves in $\Omega$)

and $\mathrm{Ind}_\Gamma(a) = 0$ for all $a$ in $\mathbb{C} - \Omega$.

Say $\Gamma$ is "homologous to zero" in $\Omega$.

Write $\Gamma \sim 0$ or $\Gamma \sim_\Omega 0$.

Feeling: "Deform" $\Gamma$ to a pt in $\Omega$

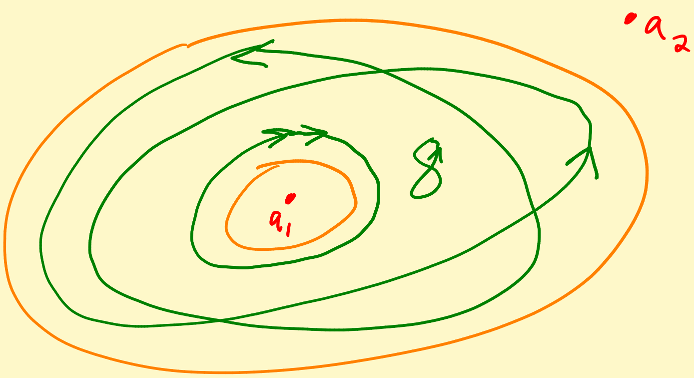


General Cauchy Theorem $\Omega$ domain in $\mathbb{C}$, $\Gamma$ is a cycle in $\Omega$, $\Gamma \sim 0$, $f$ analytic on $\Omega$. Then

1) $\displaystyle\int_\Gamma f\, dz = 0$

2) $\displaystyle f(a)\,\mathrm{Ind}_\Gamma(a) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-a}\,dz, \quad a \in \Omega - \mathrm{tr}(\Gamma).$

EX




$a_1$ : $\sum$ indices $= +2 - 2 + 0$

$a_2$ : $\sum$ ind $= 0 + 0 + 0$

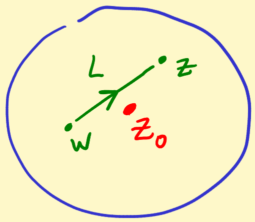
Pf (Ahlfors: due to Beardon.)

Trick(s) $\quad g(z,w) = \begin{cases} \dfrac{f(z)-f(w)}{z-w} & z \neq w \\ f'(z) & z = w \end{cases}$

Step 1 $\quad g$ is continuous on $\Omega \times \Omega \to \mathbb{C}$

Clear at $(z_0, w_0)$, $z_0 \neq w_0$. Only need to check at $(z_0, z_0)$

$\bar{D} \subset \Omega$




$$g(z,w) - g(z_0, z_0)$$

$$= \frac{f(z)-f(w)}{z-w} - f'(z_0)$$

const. $\int = f'(z_0)(z-w)$

$$= \frac{\int_L f' \, d\zeta}{z-w} - \frac{\int_L f'(z_0)\, d\zeta}{z-w}$$

So $$|g(z,w) - g(z_0,z_0)| = \frac{1}{|z-w|} \left| \int_L \left(f'(\zeta) - f'(z_0)\right) d\zeta \right|$$

$$\leq \frac{1}{|z-w|} \underbrace{\max_L |f'(\zeta) - f'(z_0)|}_{< \varepsilon} \underbrace{\text{Length}(\Gamma)}_{= |z-w|}$$

when $D$ is small
by continuity of $f'$

$\to 0$ as radius of $D \to 0$.

Step 2 $\quad$ Define $\quad h(z) = \displaystyle\int_\Gamma g(z,w)\, dw$

[Will show $h$ extends to be entire. Bndd entire.
Const. $\equiv 0$.]

Step 3 $\quad h$ is analytic on $\Omega$. Morera's!