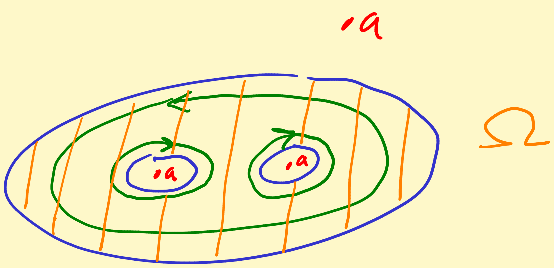
Lecture 39 Proof of the general Cauchy theorem

<u>Thm</u> Γ a cycle in a domain Ω, f analytic on Ω, $\Gamma \sim 0$ (meaning $\mathrm{Ind}_\Gamma(a) = 0$ for all $a \in \mathbb{C} - \Omega$), then

a) $\int_\Gamma f \, dz = 0$

b) $f(a)\,\mathrm{Ind}_\Gamma(a) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-a}\, dz$, $\quad a \in \Omega - \mathrm{tr}(\Gamma)$



$\Gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$

<u>Pf</u> Def $g(z,w) = \begin{cases} \dfrac{f(z)-f(w)}{z-w} & z \neq w \\ f'(z) & z = w \end{cases}$

<u>Step 1</u> g is continuous on $\Omega \times \Omega$. (Last time)

<u>Step 2</u> Define $h(z) = \int_\Gamma g(z,w)\, dw$

<u>Step 3</u> Show h analytic on Ω.

a) Clear that h is <u>continuous</u> on Ω

[why. Know g cont on $\Omega \times \Omega$. $\mathrm{tr}(\Gamma) = \bigcup_{k=1}^{N} \mathrm{tr}(\gamma_k)$ is <u>compact</u>. Suppose $z_0 \in \Omega$. Ω open. So $\exists\ \overline{D_\rho(z_0)} \subset \Omega$, $\rho > 0$, g continuous on compact $\overline{D_\rho(z_0)} \times \mathrm{tr}(\Gamma)$, so <u>uniformly</u> cont there. Rest easy. (Let $\varepsilon > 0$ ...)]

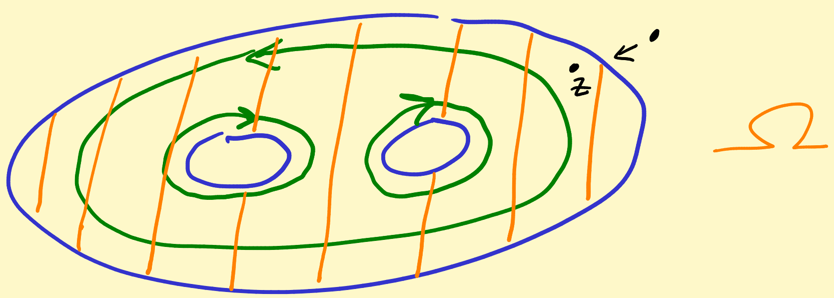
b) Apply Morera! $\Delta \subset \Omega$ triangle.

$$\int_\Delta h\,dz = \int_\Delta \left(\int_\Gamma g(z,w)\,dw \right) dz$$

$$\underset{\text{Fubinni!}}{=} \int_\Gamma \underbrace{\left(\int_\Delta g(z,w)\,dz \right)}_{=0} dw = 0 \checkmark$$

g has a removable sing in z at z=w (w fixed).

$$g(z,w) = \begin{cases} \dfrac{f(z)-f(w)}{z-w} & z \neq w \\ f'(w) & z = w \end{cases}$$



$$h(z) = \int_\Gamma \frac{f(z)-f(w)}{z-w}\,dw = f(z)\underbrace{\int_\Gamma \frac{1}{z-w}\,dw}_{=0} - \int_\Gamma \frac{f(w)}{z-w}\,dw$$

when z outside Ω. ($\Gamma \sim 0$)

Aha! Define $\tilde{h}(z) = -\int_\Gamma \frac{f(w)}{z-w}\,dw$ on

$$\tilde{\Omega} = \{ z \in \mathbb{C} - tr(\Gamma) \text{ where } Ind_\Gamma(a) = 0 \}$$

Facts: $\mathbb{C} - \mathrm{tr}(\Gamma)$ is open and is a union of connected components. Since Ind_Γ is constant on components, see $\tilde{\Omega}$ = Union of open connected components.

1) so $\tilde{\Omega}$ is open.

2) $\Omega \cup \tilde{\Omega} = \mathbb{C}$ because $\Gamma \sim 0$.

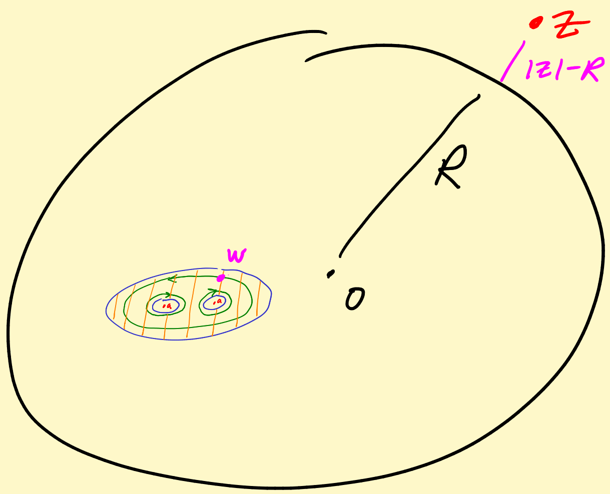
3) $\tilde{h}$ is analytic on $\tilde{\Omega}$ (HWK 2: #1)

4) $h = \tilde{h}$ on $\Omega \cap \tilde{\Omega}$ (saw from def^n above)

So $\tilde{h}$ extends h to $\mathbb{C}$ as an entire fcn!

Call extension h from now on.

Claim $h \to 0$ at ∞.



$\mathrm{Ind}_\Gamma(z) = 0$ when z outside big disc $D_R(0)$

$$|h(z)| = \left| -\int_\Gamma \frac{f(w)}{z-w}\,dw \right| \le \left(\max_\Gamma |f|\right) \frac{1}{|z|-R} \cdot \mathrm{Length}(\Gamma)$$

$$\longrightarrow 0 \text{ as } z \to \infty. \checkmark$$

Liouville's $\Rightarrow h \equiv 0$ on $\mathbb{C}$.

Done $\frac{1}{2\pi i} \int_\Gamma \frac{f(z) - f(w)}{z - w}\,dw$ $\quad z \in \mathbb{C} - \mathrm{tr}(\gamma)$

$= 0$

So $f(z) \cdot \frac{1}{2\pi i} \int_\Gamma \frac{1}{z-w}\, dw = \frac{1}{2\pi i} \int \frac{f(w)}{z-w}\, dw$ ✓

(Take − both sides). Cauchy integral formula ✓

CIF ⟹ Cauchy theorem. (Easier direction)

<u>Pf</u>: f analytic on Ω. Pick $a \in \Omega - tr(\Gamma)$.

Let $F(z) = (z-a) f(z)$

CIF for F at a:

$$\underbrace{F(a)}_{=0} \, \mathrm{Ind}_\Gamma(a) = \frac{1}{2\pi i} \int_\Gamma \frac{\overbrace{(z-a) f(z)}^{F(z)}}{z-a}\, dz$$

$$0 = \int_\Gamma f \, dz$$ ✓