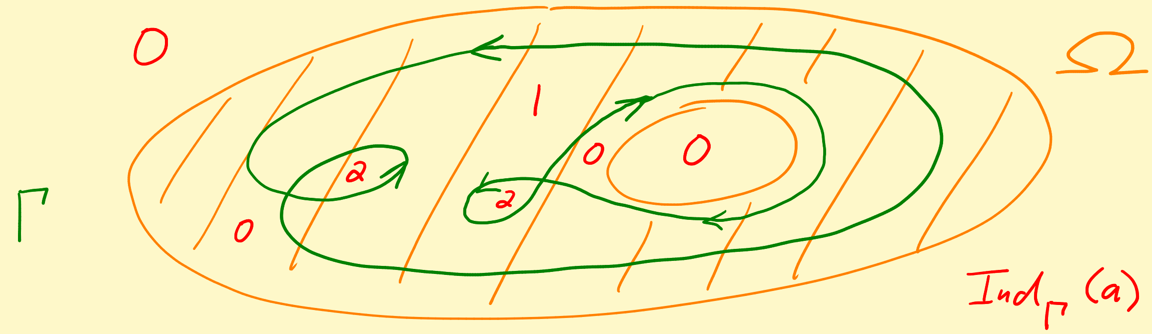
Lecture 40 General Residue theorem and Rouché's thm



Cauchy thm $\Gamma \sim 0$ in domain Ω, f analytic on Ω

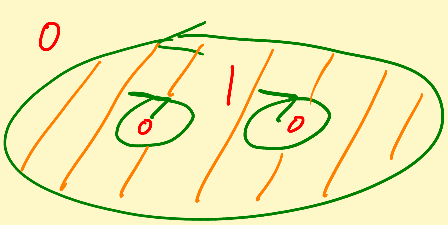
a) $\int_\Gamma f\, dz = 0$

b) $f(a)\,\mathrm{Ind}_\Gamma(a) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-a}\,dz$, $a \notin tr(\gamma)$

Words Def^n A fcn f is called meromorphic on a domain Ω if there is a discreet subset $A \subset \Omega$ (meaning a set with no limit pts in Ω) such that f is analytic on $\Omega - A$ and has poles at pts in A.

Def^n We say f is analytic on a compact set K provided there is an open set U containing K and an analytic F on U such that $F|_K = f$.

Def^n Say a cycle Γ is simple if $\mathrm{Ind}_\Gamma(z)$ is either 0 or 1 for all $z \in \mathbb{C} - tr(\Gamma)$.



▨ = "inside of Γ"

Rest = "outside"

Note $tr(\Gamma) \cup \{z : \mathrm{Ind}_\Gamma(z) \neq 0\}$ is closed and bounded, hence compact.

<u>Cauchy thm</u> If f is analytic "inside" and on a simple cycle, then a) $\int_\Gamma f\,dz = 0$

b) $f(a) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-a}\,dz$, a "inside" Γ.

<u>New improved Cauchy thm on simply connected domains</u>

Suppose γ is a closed curve in a s.c. domain Ω.

$\gamma \sim \{z_0\} \leftarrow$ γ homotopic to a pt curve: $z(t) \equiv z_0$.

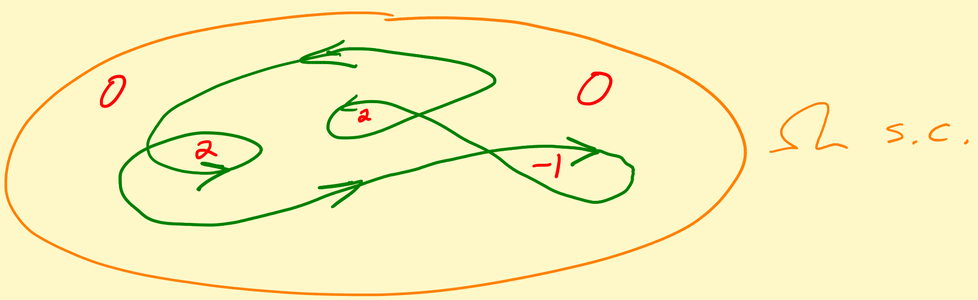
Know a) $\int_\gamma f\,dz = 0$, f analytic on Ω.

Think of $\Gamma = \gamma$ as a cycle. $\gamma \sim 0 \leftarrow$ homologous to zero

because $\mathrm{Ind}_\gamma(a) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}\,dz = 0$ when $a \in \mathbb{C} - \Omega$

via Cauchy thm (a).

New thing b) $f(a)\,\mathrm{Ind}_\gamma(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz$, $a \in \Omega - tr(\gamma)$



<u>General residue thm</u> If f is meromorphic on a domain Ω and Γ cycle in Ω, $\Gamma \sim 0$, f has no poles

on $tr(\Gamma)$, then

$$\int_\Gamma f\,dz = 2\pi i \sum_{a\in A}{}' \mathrm{Ind}_\Gamma(a)\, \mathrm{Res}_a f$$

finite sum.

$A =$ poles of f in Ω.

<u>Remark</u> Since $K = tr(\Gamma) \cup \{z : \mathrm{Ind}_\Gamma(z) \neq 0\}$ is a compact subset of Ω and A is discreet, $K \cap A$ is finite because it can't have a limit in $K \subset \Omega$.

<u>Def^n</u> $n_j C_\varepsilon(a_j)$: $z(t) = a_j + \varepsilon e^{i n_j t}$

Go around $C_\varepsilon(a_j)$ n_j times

$\begin{cases} n_j < 0 & \text{Counterclockwise} \\ n_j > 0 & \text{Clockwise} \end{cases}$

$n_j \in \mathbb{Z}$.

Easy fact $$\int_{n_j C_\varepsilon(a_j)} f\,dz = n_j \underbrace{\int_{C_\varepsilon(a_j)} f\,dz}_{= 2\pi i\, \mathrm{Res}_{a_j} f}.$$

when a_j is pole of f.

<u>Pf of Res thm</u> Let $\mathcal{O} = \{a \in A : \mathrm{Ind}_\Gamma(a) \neq 0\}$. Finite set.

Can take $\varepsilon > 0$ so small that

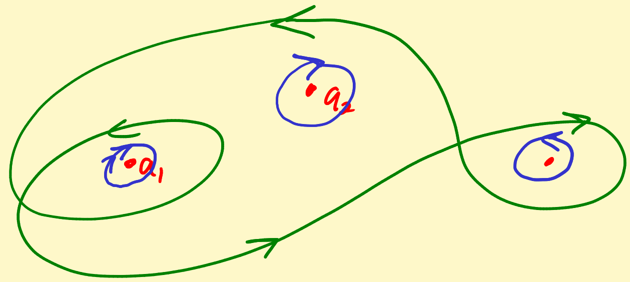
1) $\overline{D_\varepsilon(a)} \subset \Omega$ for $a \in \mathcal{O}$

2) $\overline{D_\varepsilon(a)} \cap tr(\Gamma) = \emptyset$

3) $\overline{D_\varepsilon(a_1)} \cap \overline{D_\varepsilon(a_2)} = \emptyset$, $a_1, a_2 \in \mathcal{O}$, $a_1 \neq a_2$.

Aha! $\tilde{\Omega} = \Omega - \mathcal{A}$. [Remove pts A-$\mathcal{A}$ from Ω as Step 1.]

$$\tilde{\Gamma} = \Gamma \cup \left[\bigcup_{a \in \mathcal{A}} -n_a C_\varepsilon(a) \right], \quad n_a = \mathrm{Ind}_\Gamma(a)$$



$\mathrm{Ind}_{\tilde{\Gamma}}(a) = 0$ for all a outside Ω (by Cauchy thm).

$\mathrm{Ind}_{\tilde{\Gamma}}(a) = 0$ for $a \in \mathcal{A}$ too by construction.

$$\mathrm{Ind}_{\tilde{\Gamma}}(a) = \underbrace{\mathrm{Ind}_\Gamma(a)}_{=0,\ a \text{ outside } \Omega} + \sum_{a_j \in \mathcal{A}} n_{a_j} \underbrace{\mathrm{Ind}_{C_\varepsilon(a_j)}(a)}_{=0,\ a \text{ outside } \Omega \text{ too}}$$

General Cauchy thm: $\displaystyle 0 = \int_{\tilde{\Gamma}} f\, dz = \int_\Gamma f\, dz + 2\pi i \sum_{a \in \mathcal{A}} (-n_a) \operatorname{Res}_a f$

General Argument princ and general Rouché's follow:

$$\operatorname{Res}_a \frac{f'}{f} = \begin{cases} \text{order of zero of } f \text{ at } a \\ -\text{order of pole of } f \text{ at } a. \end{cases}$$

Remark General Rouché's makes most sense when Γ is a simple cycle: $|f(z) - g(z)| < |f(z)|$ on a simple cycle, f and g analytic inside and on cycle, then they have same # zeroes "inside" cycle.

Final Exam Tuesday, Dec 10, 8:00–10:00 am in HAAS G066.

No office hr Thurs 12/5 or Mon 12/9

Special office hr Fri 12/6 after class.

Practice probs and old exam on home page.