Lecture 41 Mittag-Lefler & Weierstraß theorems

Remark General residue theorem works exactly the same for f analytic on $\Omega - A$ where A is a discreet subset of Ω (instead of f meromorphic), so essential singularities are allowed.

Suppose a_n distinct points in $\mathbb{C}$ and $a_n \to \infty$ as $n \to \infty$.

M-L Thm Given principal parts $R_n(z) = \sum_{k=1}^{N_n} \frac{A_{nk}}{(z-a_n)^k}$, there is a meromorphic fcn on $\mathbb{C}$ with poles only at the a_n having principle part $R_n(z)$ at a_n for each n.

W Thm Given positive integers m_n, there is an entire fcn with zeroes of order m_n at each a_n and no others.

Cor Meromorphic fcns on $\mathbb{C}$ $= \{f/g : f, g \text{ entire}, g \not\equiv 0\}$

Pf of Cor: Mero fcn F with poles at a_n's of order M_n.

$W \Rightarrow \exists$ entire g with zeroes of order M_n at a_n.

$f = Fg$ has removable sing at each a_n. $F = f/g$ ✓

Pf of M-L Let $f = \sum_{n=1}^{\infty} \left[R_n(z) - p_n(z) \right]$

where $p_n(z)$ is Taylor poly for $R_n(z)$ about $z=0$ such that $|R_n(z) - p_n(z)| < \frac{1}{2^n}$ on $\overline{D_{r_n}(0)}$ where $r_n = \frac{|a_n|}{2}$. Claim: f solves M-L prob.

Take big disc $D_R(0)$. $\exists$ N such that $r_n = \frac{|a_n|}{2} > R$ if $n > N$. On $D_R(0)$:

$$f(z) = \underbrace{\sum_{n=1}^{N} \left(R_n(z) - p_n(z) \right)}_{\text{Rational fcn with prescribed princ parts in } D_R(0)} + \sum_{n=N+1}^{\infty} \underbrace{\left(R_n(z) - p_n(z) \right)}$$

$|R_n - p_n| < \frac{1}{2^n}$ on $D_R(0)$. Weierstraß M-test $\Rightarrow \sum$ conv unif to an analytic fcn on $D_R(0)$.

So $f(z) = R_n(z) + (\text{analytic fcn})$ near a_n in $D_R(0)$. ✓

Remark M-L Thm true for discreet A in any domain Ω. Can cook up mero with prescribed poles at pts in A.

Pf of W thm M-L + Gen^l residue thm $\Rightarrow$ W thm.

Hmmm. $\frac{f'}{f} = \frac{m_n}{z - a_n} + (\text{analytic near } a_n)$

M-L Thm $\Rightarrow \exists$ mero F with poles at a_n's with princ parts $\frac{m_n}{z-a_n}$

Hmmm. $f(z)$ "=" $e^{\log f} = \exp\left(\int_{\gamma_a^z} \frac{f'}{f}\, dw \right)$

Details: M-L $\Rightarrow \exists$ mero F on $\mathbb{C}$ with poles at a_n's with princ parts $\frac{m_n}{z-a_n}$

"Define" $f(z) = \exp\left(\int_{\gamma_a^z} F \, dw \right)$ where γ_a^z is a curve in $\mathbb{C} - \{a_n\}_{n=1}^{\infty}$ starting at a fixed pt a in the set and ending at z.

<u>Step 1</u> $f(z)$ is <u>well</u> defined.

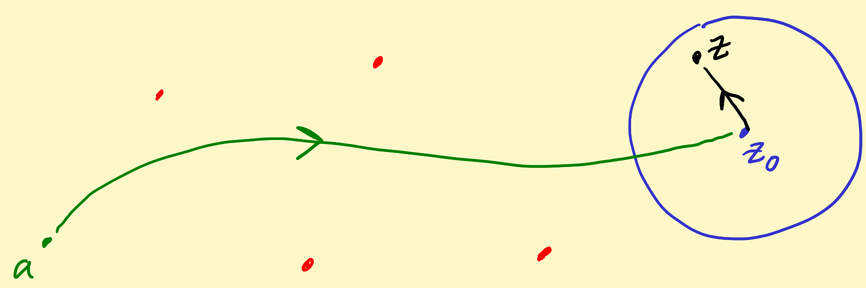
$$\frac{f(z)}{\tilde{f}(z)} = \exp\left(\underbrace{\left[\int_{\gamma_a^z} - \int_{\tilde{\gamma}_a^z} \right]}_{\Gamma = \gamma_a^z \cup (-\tilde{\gamma}_a^z) \text{ is closed!}} F \, dw \right)$$

$$= \exp\left(2\pi i \underset{\substack{\uparrow \\ \text{finite } \Sigma}}{\sum} \underbrace{\mathrm{Ind}_\Gamma(a_n)}_{\text{integer}} \underbrace{\mathrm{Res}_{a_n} F}_{m_n} \right)$$

$$= e^{k(2\pi i)}, \quad k \text{ an integer.} \quad = 1.$$

So $f(z) = \tilde{f}(z)$ is indep of choice of γ_a^z.

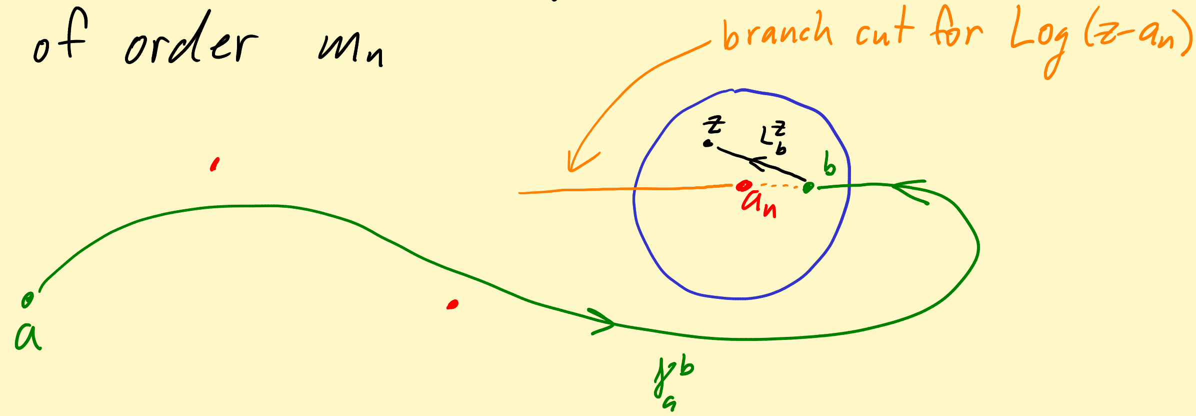
<u>Step 2</u> f is analytic on $\mathbb{C} - \{a_n\}_1^\infty$.



$$f(z) = \exp\left[\underbrace{\int_{\gamma_a^{z_0}} F \, dw}_{\text{const}} + \underbrace{\int_{L_{z_0}^z} F \, dw}_{\text{anti-deriv for } F \text{ on disc.}} \right]$$

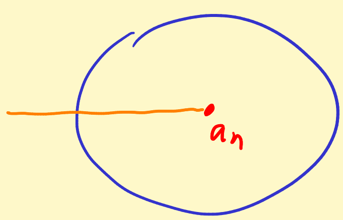
is analytic on disc.

Step 3 a_n is a removable sing for f and is in fact a zero of order m_n



$$f(z) = \exp\left(\underbrace{\int_{\gamma_a^b} F\,dw}_{\text{const}} + \underbrace{\int_{L_b^z} \frac{m_n}{w-a_n}\,dw}_{m_n\left(\mathrm{Log}(z-a_n) - \underbrace{\mathrm{Log}(b-a_n)}_{\text{const}}\right)} + \underbrace{\int_{L_b^z} \underbrace{H(w)}_{\substack{\text{analytic}\\ \text{on disc}}}\,dw}_{\substack{\text{analytic}\\ \text{anti-deriv of } H}} \right)$$

Aha! $$f(z) = \underbrace{\exp\left(m_n \mathrm{Log}(z-a_n)\right)}_{=(z-a_n)^{m_n}} \cdot \underbrace{\exp\left(\text{analytic fcn on disc}\right)}_{\substack{\text{analytic on disc}\\ \text{and non-vanishing there}}}$$



fcn analytic above and below cut and extends continuously up to cut.

Morera argument $\Rightarrow$ cut removable.

Weierstraß used infinite products to construct f. Beautiful math!

Stein: p. 140–153

Ahlfors: p. 191–197