Review

Final Exam Tues 8:00–10:00 am HAAS G066

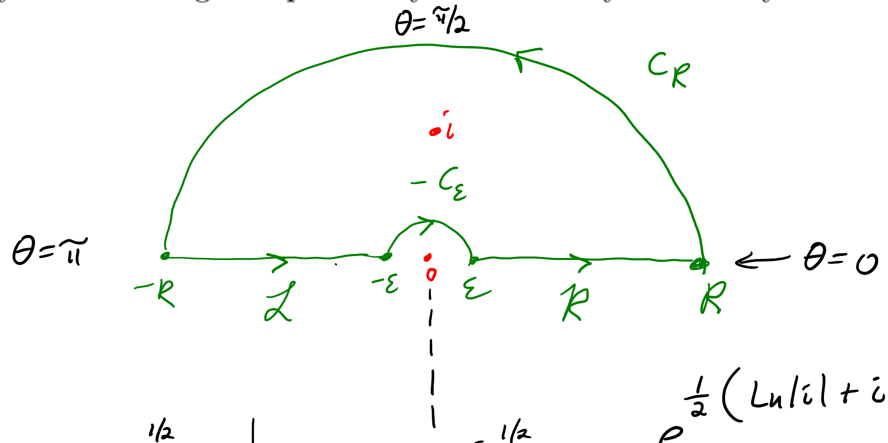
Closed book, no notes or electronics. Just need pencils/eraser.

1. Calculate

$$\int_0^\infty \frac{\sqrt{x}}{(x^2+1)}\,dx.$$

$f(z) = \dfrac{z^{1/2}}{z^2+1}$ $\leftarrow$ $z^{1/2} = e^{\frac{1}{2}\log z}$ where $\log z = \ln|z| + i\theta$, $\theta \in \arg z$, $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$

Explain your reasoning and prove any limits that you use in your work.



$$\operatorname{Res}_i f(z) = \left.\frac{z^{1/2}}{\frac{d}{dz}(z^2+1)}\right|_{z=i} = \frac{i^{1/2}}{2i} = \frac{e^{\frac{1}{2}\left(\ln|i| + i\frac{\pi}{2}\right)}}{2i} = \frac{e^{i\pi/4}}{2i}$$

$|z^{1/2}| = |z|^{1/2}$. Show $\int_{C_{-\varepsilon}}, \int_{C_R} \to$ as $\varepsilon \searrow 0$ and $R \to \infty$.

$-\mathcal{L}$: $z(t) = -t$, $\varepsilon \le t \le R$ $\quad z'(t) = -1$

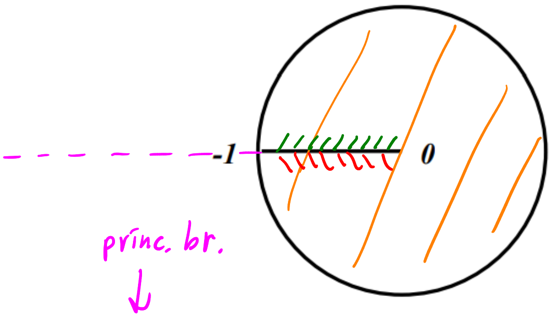
$$\int_{\mathcal{L}} f(z)\,dz = -\int_{-\mathcal{L}} f(z)\,dz = -\int_\varepsilon^R \frac{(-t)^{1/2}}{[(-t)^2+1]}(-1)\,dt$$

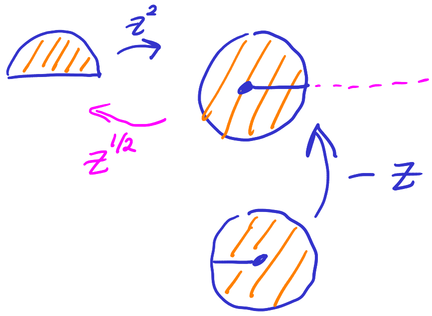
$$= \int_\varepsilon^R \frac{e^{\frac{1}{2}(\ln t + i\pi)}}{t^2+1}\,dt$$

$$= \underbrace{e^{i\pi/2}}_{=i} \int_\varepsilon^R \frac{\sqrt{t}}{t^2+1}\,dt$$

etc.

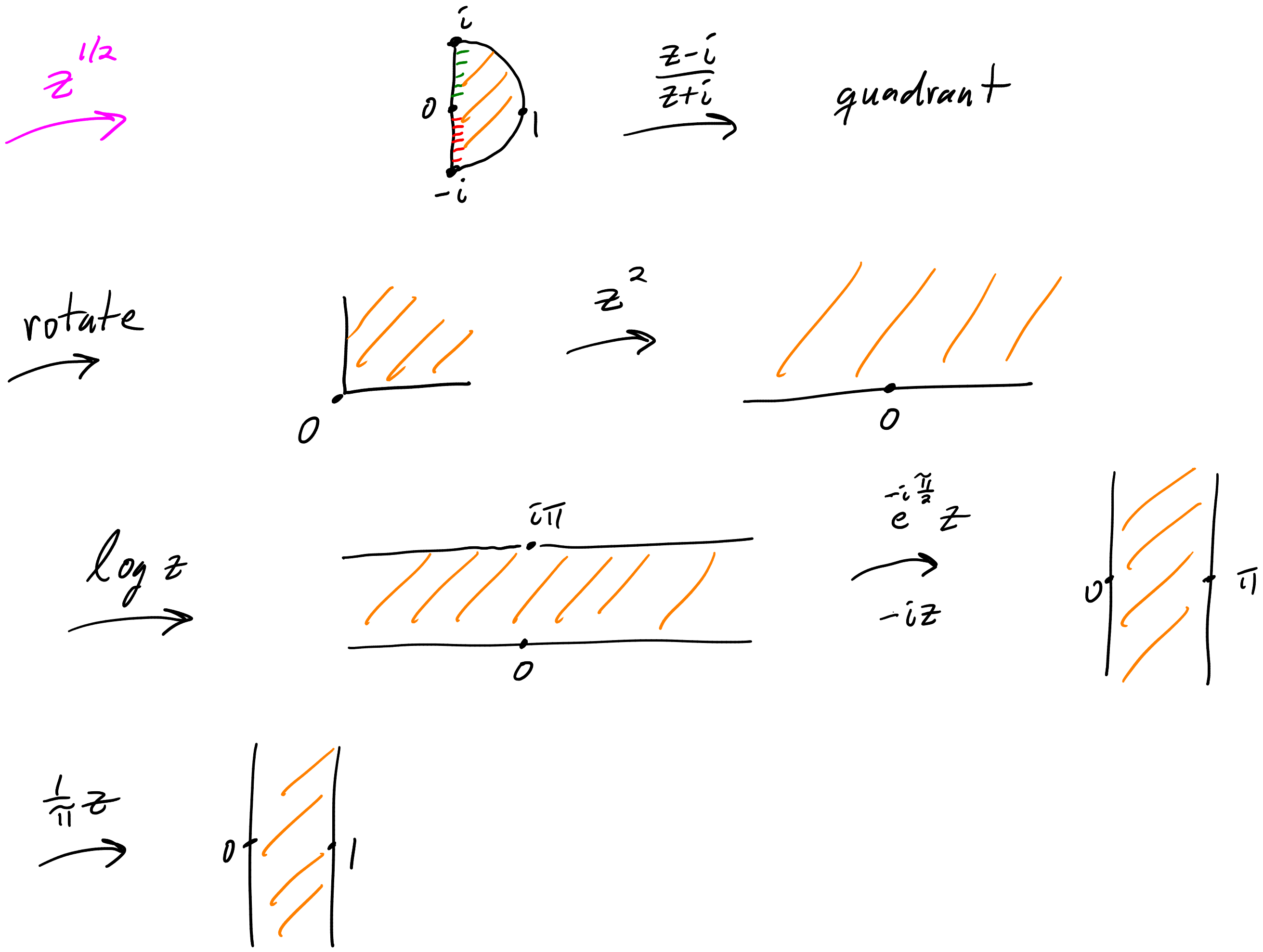
2. Let Ω denote the slit disk $D_1(0) - (-1, 0]$. Find a one-to-one conformal mapping of Ω onto the vertical strip $\{z : 0 < \text{Re } z < 1\}$. (Express your answer as a composition of maps, but don't compute the composition.)



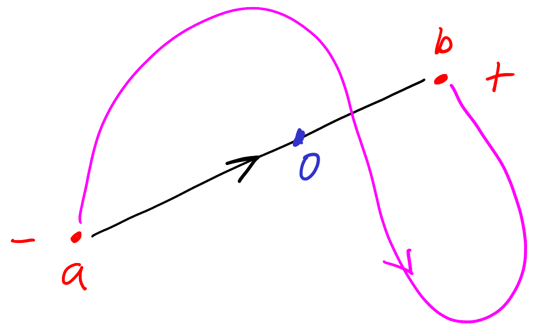


princ. br.

$$z^{1/2} = e^{\frac{1}{2}\,\mathrm{Log}\, z} = \underbrace{e^{\frac{1}{2}\mathrm{Ln}|z|}}_{\sqrt{|z|}}\, e^{i(\mathrm{Arg}\, z)/2}$$



3. Prove that a real valued function $\overset{u}{v}$ that is harmonic on the whole complex plane and that has no zeroes must be constant.



$u(z(t))$

$z(t) = a + t(b-a)$

$0 \le t \le 1$

Let v be a harm conj for u on $\mathbb{C}$. $f = u + iv$ entire

$u > 0$ case : $f : \mathbb{C} \longrightarrow RHP$. Casorati–Weierstraß $f \equiv \text{const}$.

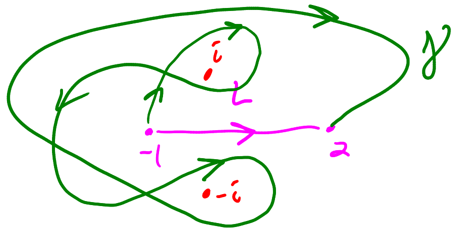
$u < 0$ case : Replace f by $-f$. use above

<u>or</u> $f : \mathbb{C} \to LHP$. C-W thm.

<u>Or</u> $e^f = e^{u+iv} = e^u e^{iv}$

$|e^f| = e^u$. Use Liouville's for $e^{\pm f}$, entire.

5. What are the possible values of the integral



$$\int_\gamma \frac{z}{z^2+1}\,dz,$$

where γ is a path that starts at $z=-1$ and ends at $z=2$ without passing through the points $\pm i$. Explain.

$$\int_L f(z)\,dz = \int_{-1}^{2} \frac{t}{t^2+1}\,dt = \frac{1}{2}\operatorname{Ln}(t^2+1)\Big|_{-1}^{2} = \frac{1}{2}\operatorname{Ln}5 - \frac{1}{2}\operatorname{Ln}2 = \operatorname{Ln}\sqrt{5/2}$$

Key: $\Gamma = L \cup \{-\gamma\}$ closed curve, missing $\pm i$.

$\Omega = \mathbb{C}$

General residue thm: $\underbrace{\int_\Gamma f(z)\,dz}_{\int_L + \int_{-\gamma}} = 2\pi i \sum_{\text{two}} \underbrace{\operatorname{Ind}_\Gamma(\pm i)}_{\text{integers}} \operatorname{Res}_{\pm i} f(z)$

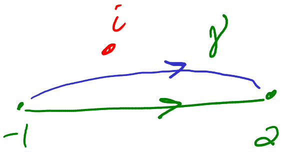
$$\operatorname{Res}_{\pm i} f(z) = \frac{\pm i}{2(\pm i)} = \frac{1}{2}$$

$$\int_\gamma f\,dz = \int_L - 2\pi i\left(n_1\cdot\frac{1}{2} + n_2\cdot\frac{1}{2}\right)$$

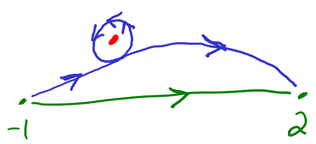
$$= \operatorname{Ln}\sqrt{5/2} - \pi i\underbrace{(n_1+n_2)}_{m\in\mathbb{Z}} \overset{?}{=} \operatorname{Ln}\sqrt{5/2} + \pi i k$$

any $k\in\mathbb{Z}$ possible?

$k=0$:

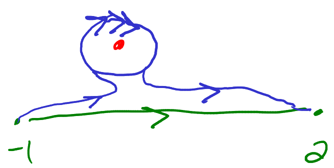


$k=3$:



Any $k\in\mathbb{Z}$ possible

$k=-3$:



2. Suppose that A is a finite set and that $f(z)$ is analytic on $\mathbb{C} - A$ with poles at each point in A. Prove that if f has a removable singularity at infinity, then f must be a rational function.

$|f(z)| < M$

for $|z| > R$.

Hint: Partial fracs.

$A = \{a_j : j = 1, \ldots, N\}$

$$F(z) = f(z) - \sum_{j=1}^{N} \underbrace{R_j(z)}$$

Princ part f at a_j

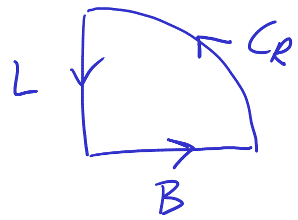
F has removable sing at each a_j, so entire.

$R_j \to 0$ at ∞. F bndd entire. Liouville's.

4. How many zeroes does the polynomial

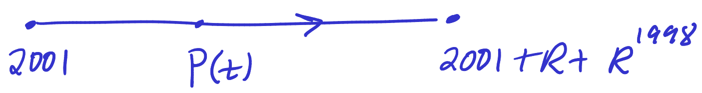
$$z^{1998} + z + 2001$$

have in the first quadrant? Explain your answer.



$$\# \text{zeroes } P = \Delta_\gamma \arg f(z)$$

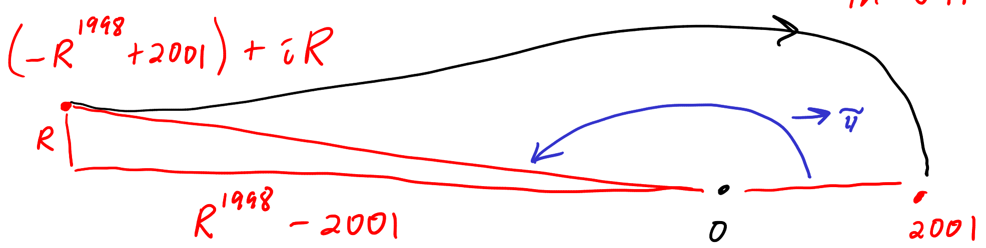
B :



$$\Delta \arg = 0$$

$$-L : \quad (it) : \quad 0 \le t \le R$$

$$P(it) = \left(-t^{1998} + 2001\right) + \underbrace{it}_{\text{in UHP}}$$



$$\Delta \arg \to -\pi$$

C_R

$$\Delta_{C_R} \operatorname{Arg} P(z) = \operatorname{Im} \int_{C_R} \frac{P'(z)}{P(z)} \, dz$$

$$\frac{P'(z)}{P(z)} = \frac{1998 z^{1997} + 1}{z^{1998} + z + 2001} \sim \frac{1998}{z}$$

$$\frac{P'(z)}{P(z)} - \frac{1998}{z} = \text{cross mult} = \frac{\text{poly deg } 1}{\text{poly deg } 1999}$$

$$\int_{C_R} \frac{P'(z)}{P(z)} - \frac{1998}{z} \, dz \to 0 \quad \text{as } R \to \infty.$$

499 zeroes : ans.