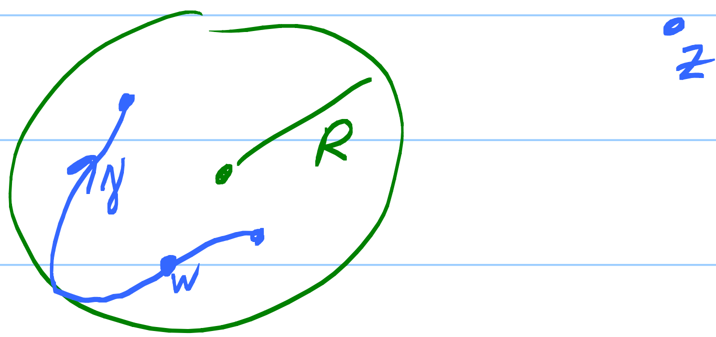
Solutions to Exam 1

1. Suppose $f(x+iy) = u(x,y) + i\underbrace{v(x,y)}^{\equiv 0}$ is a real valued analytic function a domain Ω. The Cauchy-Riemann Equations yield $u_x = v_y \equiv 0$ and $u_y = -v_x \equiv 0$; so $\nabla u \equiv 0$ on Ω. (Note that f analytic $\Rightarrow$ u C^∞-smooth.) Ω connected open $\Rightarrow$ Ω path connected. Given points a and z in Ω, let γ be a curve joining a to z. $0 = \int_\gamma \nabla u \cdot d\vec{s} = u(z) - u(a)$. Fixing a and letting z vary shows that $u(z) \equiv u(a)$ on Ω.

2. Notice that $\frac{2+n}{3+n^2} = \frac{n(\frac{2}{n}+1)}{n^2(\frac{3}{n^2}+1)} = \frac{\frac{1}{n}(2\cdot\frac{1}{n}+1)}{(3\cdot\frac{1}{n^2}+1)}$.
Let $g(z) = \frac{z(2z+1)}{3z^2+1}$, which is analytic on $D_{1/\sqrt{3}}(0)$. If such an f existed, we see that $f(\frac{1}{n}) = g(\frac{1}{n})$ for $n = 2, 3, \ldots$. The set $\{\frac{1}{n} : n = 2, 3, \ldots\}$ has a limit point in $D_{1/\sqrt{3}}(0)$. The Identity Theorem implies that $f \equiv g$ on $D_{1/\sqrt{3}}(0)$. But g blows up at $\pm\frac{i}{\sqrt{3}}$ and f does not. So f cannot be

analytic on $D_1(0)$.

③ Let $R = \text{Max}\{|z(t)| : a \le t \le b\}$ where $z(t)$ is a parametrization for γ.



If $w \in tr(\gamma)$ and $|z| > R$, then $|w - z| \ge \big||w| - |z|\big|$ $= |z| - |w| \ge |z| - R$. Hence,

$$|F(z)| = \left| \int_\gamma \frac{\varphi(w)}{w-z}\, dw \right| \le \left(\underset{\gamma}{\text{Max}} |\varphi| \right) \cdot \frac{1}{|z|-R} \cdot \text{Length}(\gamma)$$

and this tends to zero as $z \to \infty$.

④ Suppose g has an essential singularity at $z=0$. Then g is not constant and its zeroes on $D_1(0) - \{0\}$ are isolated. Let $h(z) = \frac{f(z)}{g(z)}$ for $z \in D_1(0) - \{0\}$ where $g(z) \neq 0$. If z_0 is an isolated zero of g on $D_1(0) - \{0\}$, notice that $|h|$ is bounded by 1 near z_0 and has an isolated singularity at z_0. The R.R.S.T. $\Longrightarrow$

that h has a removable singularity at z_0. By giving h proper values at the zeroes of g, we can view it as an analytic function on $D_1(0)-\{0\}$. Continuity of analytic functions shows that $|h|$ is bounded by 1 on $D_1(0)-\{0\}$ too. Hence the RRST yields that 0 is removable for h. Now $g=\frac{f}{h}$. If f has a removable singularity or a pole at 0, then $f(z)=z^N F(z)$ where $N\in\mathbb{Z}$ and F is analytic on $D_1(0)$ with $F(0)\neq 0$. Also $h(z)=z^M H(z)$ where $M\in\{0,1,2,\dots\}$, H analytic on $D_1(0)$ and $H(0)\neq 0$. Hence $g=z^{N-M}\left(\frac{F(z)}{H(z)}\right)$ near 0 and we see that g has a pole or a removable singularity at $z=0$. $\lightning$.

Hence, f must have an essential singularity at 0.