MA 530 Steve Bell

www.math.purdue.edu/~bell/MA530

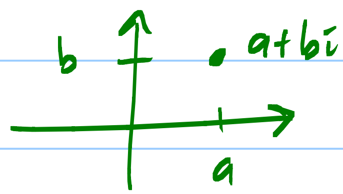
HWK 100 (Lidia Mrad)

2 in class exams 100, 100

Final Exam 200

500 points

$\mathbb{C} \approx \mathbb{R}^2$ $i^2 = -1$ $x^2 + 1 = 0$



$a + bi$

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$$
$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2 - b_1 b_2, a_1 b_2 + a_2 b_1)$$

$\mathbb{C}$ is a <u>field</u>.

$(0,0) \sim 0$

$(1,0) \sim 1$

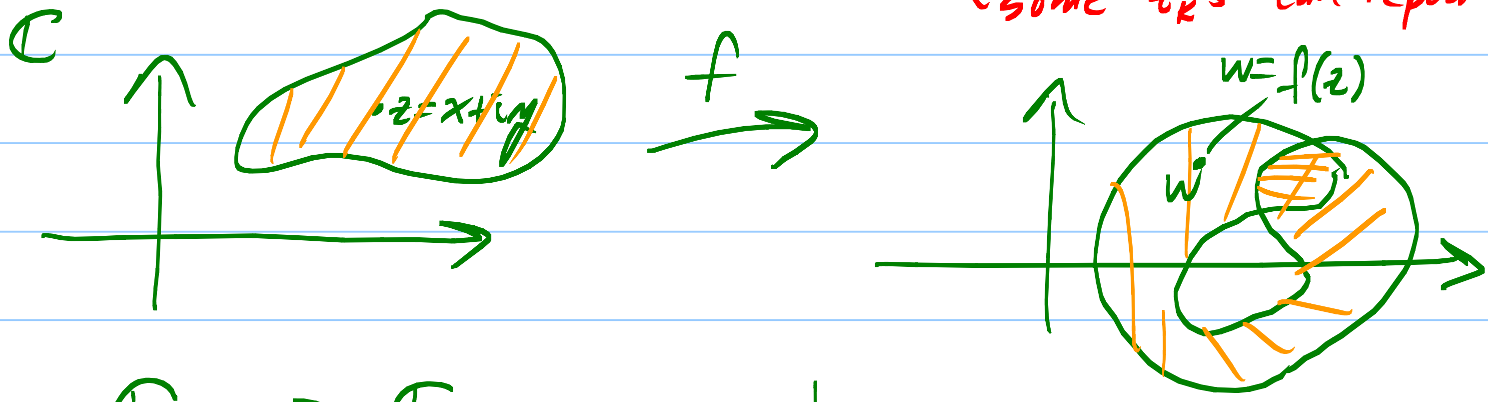
$(0,1) \sim i$

<u>Frobenius</u>: Every division ring that is a finite extension of $\mathbb{R}$ is either
1) $\approx \mathbb{R}$
2) $\approx \mathbb{C}$
3) $\approx$ Quarternions

<u>Fundamental Theorem of Algebra</u>: Given a complex polynomial $P(z) = a_N z^N + a_{N-1} z^{N-1} + \cdots + a_1 z + a_0$ of degree $N \geq 1$, then $P(z)$ has a root in $\mathbb{C}$,

i.e., $\exists\, z_0 \in \mathbb{C}$ with $P(z_0) = 0$. Consequently, can factor $P(z) = a_N \prod_{k=1}^{N} (z - z_k)$.

some z_k's can repeat



$\mathbb{C} \to \mathbb{C}$

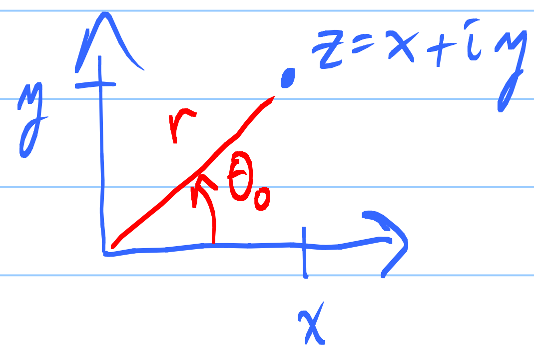
$w = f(z)$

$\underset{\substack{\uparrow \\ u(x,y)}}{u} + i \underset{\substack{\uparrow \\ v(x,y)}}{v} = f(x+iy)$

$\mathbb{R}^2 \to \mathbb{R}^2$

$(x,y) \longmapsto (u(x,y), v(x,y))$

<u>EX</u>: $f(z) = z^2 = (x+iy)^2 = \underbrace{(x^2 - y^2)}_{u} + i \underbrace{2xy}_{v}$



$x = \operatorname{Re} z$

$y = \operatorname{Im} z$

$r = |z| = \sqrt{x^2+y^2}$

modulus of z

$\bar{z} = x - iy$

conjugate of z

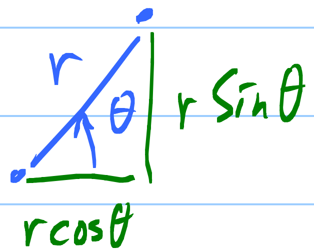
$\theta_0 = \operatorname{Arg} z \leftarrow$ Principal argument of z

$-\pi < \theta_0 \leq \pi$

$\arg z = \{\theta : \theta_0 + 2\pi n,\ n \in \mathbb{Z}\}$ ← a set

<u>Def^n</u>: $e^{i\theta} = \cos\theta + i\sin\theta$ ← $\text{cis}(\theta)$

$re^{i\theta} = (r\cos\theta) + i\,(r\sin\theta)$ ← polar form



$$(e^{i\theta})^2 = \cos 2\theta + i\sin 2\theta = e^{i2\theta}$$

$$(e^{i\theta})^n = e^{in\theta}$$

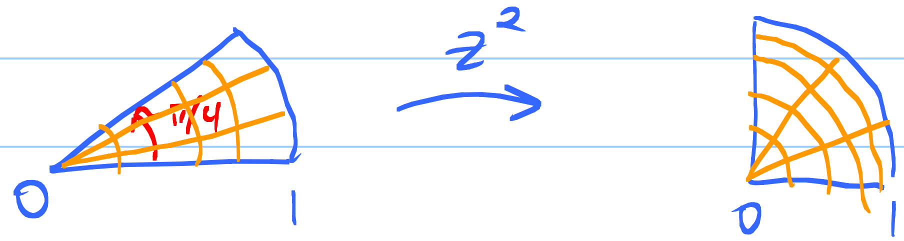
<u>Fun thing</u>: $1 + \sin\theta + \sin 2\theta + \cdots + \sin N\theta$

$$= \text{Im}\left[1 + e^{i\theta} + e^{i2\theta} + \cdots + e^{iN\theta}\right]$$

$$= \text{Im}\left[\sum_{n=0}^{N} (e^{i\theta})^n\right] = \text{Im}\left[\frac{1-(e^{i\theta})^{N+1}}{1-(e^{i\theta})}\right]$$

$$= \text{Im}\left[\frac{1-e^{i(N+1)\theta}}{1-(e^{i\theta})}\right] = \text{short and easy.}$$

EX $f(z) = z^2 = (re^{i\theta})^2 = r^2 e^{i2\theta}$

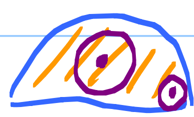


Topology on $\mathbb{C}$: $\approx \mathbb{R}^2$

$$d(z_1, z_2) = |z_1 - z_2| \leftarrow \text{distance fcn}$$

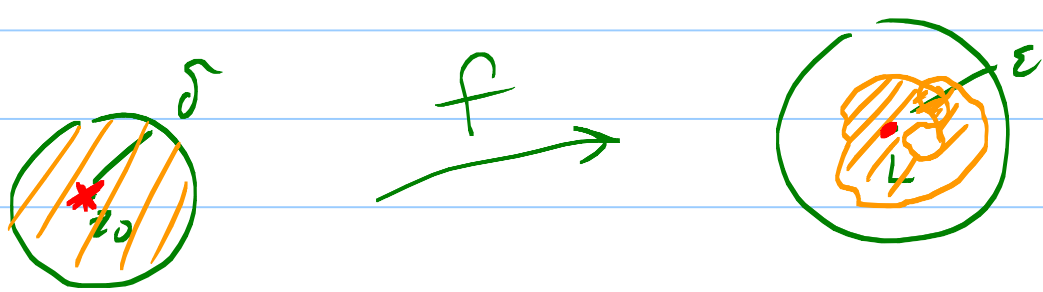
$$D_r(z_0) = \{z : |z - z_0| < r\} \leftarrow \text{disc of radius } r \text{ about } z_0$$

Def^n: $S \subset \mathbb{C}$ is open if, given any $z_0 \in S$, there is an $r > 0$ such that $D_r(z_0) \subset S$.

EX:  $= \{z : |z| < 1,\ \operatorname{Im} z > 0\}$

Def^n: Suppose f is a $\mathbb{C}$-valued fcn on $D_r(z_0) - \{z_0\}$ and $L \in \mathbb{C}$.

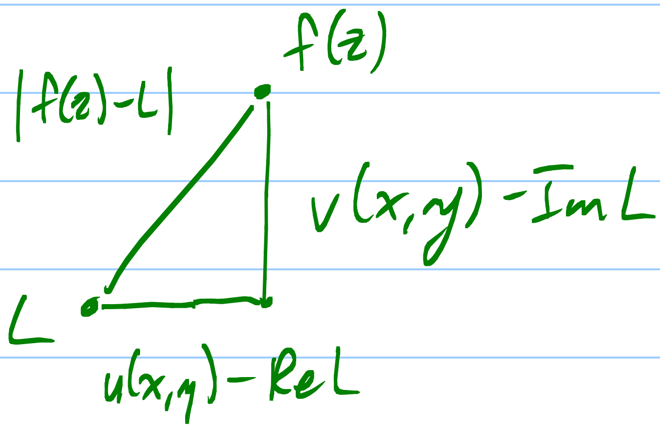
$\lim_{z \to z_0} f(z) = L$ means, given $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(z) - L| < \varepsilon$ if $|z - z_0| < \delta$ and $z \neq z_0$.



Fact: Basic laws of limits $\mathbb{R} \to \mathbb{R}$ hold $\mathbb{C} \to \mathbb{C}$ with same proofs.

Baby Theorem: $u + iv = f(x+iy)$ $\quad z_0 = x_0 + iy_0$

$$\lim_{z \to z_0} f(z) = L \iff \begin{cases} \displaystyle\lim_{(x,y) \to (x_0,y_0)} u(x,y) = \operatorname{Re} L \\ \text{and} \\ \displaystyle\lim_{(x,y) \to (x_0,y_0)} v(x,y) = \operatorname{Im} L \end{cases}$$



Def^n: Continuity: Same words as $\mathbb{R} \to \mathbb{R}$.

$$\lim_{z \to z_0} f(z) = f(z_0).$$

Complex Differentiation: Ex: $f(z) = z^2$

$$DQ = \frac{f(z) - f(a)}{z - a} = \frac{z^2 - a^2}{z - a} = \frac{(z-a)(z+a)}{z-a}$$

$$= z + a \longrightarrow 2a \quad \text{as} \quad z \to a.$$

Write: $f'(a) = 2a$.

Question: Is $f(z) = \bar{z}$ complex diff'ble?