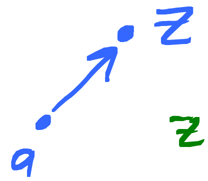
$f(z) = \bar{z}$



$z - a = re^{i\theta} = r\cos\theta + i r \sin\theta$

$$DQ = \frac{f(z)-f(a)}{z-a} = \frac{\bar{z}-\bar{a}}{z-a} = \frac{re^{-i\theta}}{re^{i\theta}} = e^{-2i\theta}$$

oops! Not complex diff'ble

Def^n: f is $\mathbb{C}$-diff'ble at z_0 if

$$\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$ exists. Call it $f'(z_0)$.

Def^n: f is analytic (or holomorphic) on an open set Ω if it is $\mathbb{C}$-diff'ble at every pt in Ω.

Thm: Analytic fcns are continuous.

Pf:
$$\frac{f(z)-f(a)}{z-a} - f'(a) = E(z) \qquad (*)$$

Note: f is $\mathbb{C}$-diff'ble at $a \iff \lim_{z \to a} E(z) = 0$

Define $E(a) = 0$.

Solve $(*)$ for $f(z)$:

$$f(z) = \underbrace{f(a) + f'(a)(z-a)}_{\mathbb{C}\text{-linear}} + \underbrace{E(z)(z-a)}_{\text{small!}}$$

$$f(z) - f(a) = f'(a)(z-a) + E(z)(z-a)$$

Let $z \to a$: $\quad f'(a) \cdot 0 + 0 \cdot 0$

So $f(z) - f(a) \to 0$ as $z \to a$, i.e., $\lim_{z \to a} f(z) = f(a)$.

Thm: $\mathbb{C}$-diff'ble at $a \implies$ cont. at a.

<u>Cauchy-Riemann Eqns</u> (C-R Eqns). Assume $f(x+iy) = u(x,y) + i\,v(x,y)$ is $\mathbb{C}$-diff'ble at $z_0 = x_0 + i y_0$. Then the first order partial derivatives of u and v exist at (x_0, y_0) and satisfy the C-R Eqns:

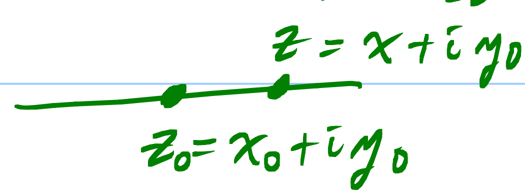
$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases} \quad \text{at } (x_0, y_0)$$

<u>Notation</u>: $\quad u_x = \dfrac{\partial u}{\partial x}$, etc.

$u^0 = u(x_0, y_0)$, etc.

$u_x^0 = u_x(x_0, y_0)$, etc.

<u>Pf</u>: <u>Step 1</u>: Slide z into z_0 along the horizontal:



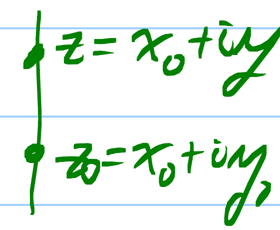
$$DQ = \frac{f(z)-f(z_0)}{z-z_0} = \frac{u(x,y_0)+iv(x,y_0)-(u^0+iv^0)}{(x+iy_0)-(x_0+iy_0)}$$

$$= \frac{u(x,y_0)-u(x_0,y_0)}{x-x_0} + i\,\frac{v(x,y_0)-v(x_0,y_0)}{x-x_0}$$

Baby Thm $\Rightarrow$ Real and Imag parts converge as $x \to x_0$. Conclude that u_x^0, v_x^0 exist.

<u>And</u> $$\boxed{f'(z_0) = u_x(x_0,y_0) + i v_x(x_0,y_0)}$$

<u>Step 2</u>: Slide in along vertical:



$$DQ = \frac{(u(x_0,y)+iv(x_0,y)) - (u^0+iv^0)}{(x_0+iy)-(x_0+iy_0)}$$

$$= \frac{v(x_0,y)-v(x_0,y_0)}{y-y_0} - i\,\frac{u(x_0,y)-u(x_0,y_0)}{y-y_0}$$

$$\longrightarrow \quad v_y(x_0,y_0) - i\,u_y(x_0,y_0) \longleftarrow$$ first partials exist.

This time, $$\boxed{f'(z_0) = v_y(x_0,y_0) - i\,u_y(x_0,y_0)}$$

Equate red boxes to get C-R Eqns.

EX: $E(x+iy) = e^{x+iy} = e^x e^{iy}$ Euler!

$= e^x(\cos y + i \sin y)$

$\underset{\text{def}^n}{=} \underbrace{e^x \cos y}_{u} + i\, \underbrace{e^x \sin y}_{v}$

Check that u and v satisfy C-R Eqns.

Question: Is $E(z)$ analytic? If so,

C-R Eqns show $E'(z) = \underbrace{u_x + i v_x}_{\text{red box \#1}} = u + iv = E(z)$

Partial Converse to C-R Eqns: Suppose u and v are C^1-smooth [u,v cont.; u_x, u_y, v_x, v_y exist and are cont.] and suppose they satisfy C-R Eqns on an open set Ω. Then $f(x+iy) = u(x,y) + iv(x,y)$ is analytic on Ω.

Looman–Menchoff Thm: Only need to suppose u,v continuous, first partials exist and satisfy C-R eqns above.

Pf: R. Narasimhan, Complex Analysis in one variables, p. 43-50.

Calculus fact: Taylor's formula in $\mathbb{R}^2$:

If u is C^1-smooth, then

$$u(x,y) = u(x_0,y_0) + u_x(x_0,y_0)(x-x_0) + u_y(x_0,y_0)(y-y_0) + R(x,y)$$

where $R(x,y) \to 0$ as $(x,y) \to (x_0,y_0)$ so fast that

$$\lim_{(x,y)\to(x_0,y_0)} \frac{R(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$$

[Pf. See MA 530 home page.]

Pf of partial converse: Assume C-R Eqns.

$$DQ = \frac{f(z)-f(z_0)}{z-z_0} = \frac{(u+iv)-(u^0+iv^0)}{z-z_0} =$$

$$\frac{1}{z-z_0}\left[u_x^0(x-x_0)+\overset{\overset{-v_x^0}{\shortparallel}}{u_y^0}(y-y_0)+i\left(v_x^0(x-x_0)+\overset{\overset{u_x^0}{\shortparallel}}{v_y^0}(y-y_0)\right)\right]$$

$$+ \frac{R_u+iR_v}{z-z_0}$$

$$\boxed{\begin{aligned} u &= u^0+u_x^0(x-x_0) + u_y^0(y-y_0)+R_u \\ v &= \cdots + R_v \end{aligned}}$$

$$\frac{1}{z-z_0}\left[\left(u_x^0+iv_x^0\right)\underbrace{\left((x-x_0)+i(y-y_0)\right)}_{z-z_0}\right]+\underbrace{\frac{R_u+iR_v}{z-z_0}}_{\varepsilon}$$

$$DQ = u_x^0 + i v_x^0 + \varepsilon$$

Calculus Fact: $|\varepsilon| \le \dfrac{|R_u|}{|z-z_0|} + \dfrac{|R_v|}{|z-z_0|} \longrightarrow 0$

as $z \to z_0$.

So $DQ \to u_x^0 + i v_x^0$. f analytic!

Cor: $E(z)$ is $\underbrace{\text{analytic on } \mathbb{C}}_{\text{entire function}}$.