Complex series

Lemma: If a series $\sum_{n=0}^{\infty} a_n$ of complex numbers a_n converges, then $a_n \to 0$ as $n \to \infty$.

Pf: Suppose $\sum_{n=0}^{\infty} a_n = L \in \mathbb{C}$. Let $s_N = \sum_{n=0}^{N} a_n$. Note that $\lim_{N\to\infty} s_N = L$ and that $a_N = s_N - s_{N-1} \longrightarrow L - L = 0$ as $n \to \infty$.

HWK1, Prob. 6. A complex power series $\sum_{n=0}^{\infty} a_n z^n$ has a radius of convergence R given by the supremum of all $r \in [0,\infty)$ such that $a_n r^n$, $n \geq 0$, is a bounded sequence.

The easy half: If $\sum_{n=0}^{\infty} a_n z^n$ converges, then $|a_n z^n| = |a_n||z|^n \to 0$ as $n \to \infty$ by the Lemma. Hence $a_n r^n$ is bounded, $r = |z|$.

The harder half: If $a_n r^n$, $r > 0$, is a bounded sequence, then $\sum_{n=0}^{\infty} a_n z^n$ converges for all $z \in \mathbb{C}$ with $|z| < r$.

Hint: Compare $\sum_{n=0}^{\infty} |a_n z^n|$ to a geometric series $\sum_{n=0}^{\infty} m\rho^n$ and use the fact that an absolutely convergent series converges (Prob. 5).

Finally, show that $\sum_{n=0}^{\infty} a_n z^n$ converges if $|z| < R$ and diverges if $|z| > R$.