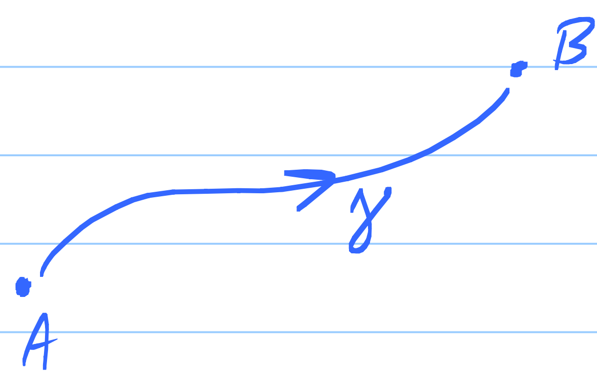
<u>Huge facts</u>: If $f(z)$ is analytic on a disc $D_r(z_0)$, then f is given by a <u>convergent power series</u> $\sum_{n=0}^{\infty} a_n(z-z_0)$ there. R. of C. $R \geq r$.

<u>And</u> convergent power series converge to analytic fcns inside their R. of C.

<u>Stein</u>: Bare hands proof of <u>And</u>, and R. of C. for f' series = R. of C. for f series.

Integration of complex fcns along curves in $\mathbb{C}$



$(x(t), y(t))$

$a \leq t \leq b$

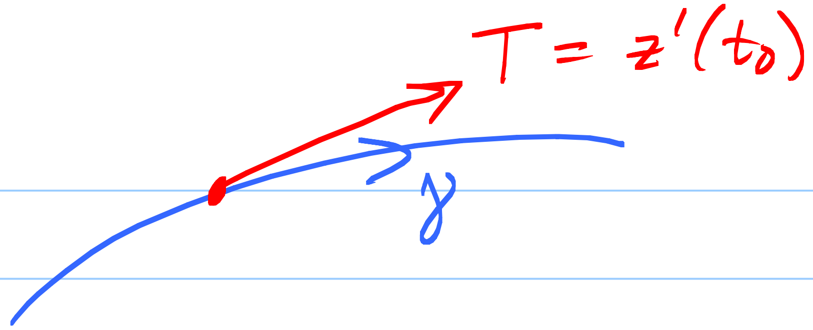
$z(t) = x(t) + iy(t) \leftarrow$ complex fcn of a real variable t

$z : [a,b] \to \mathbb{C}$

<u>Def^n</u>: $z'(t_0) = \lim_{t \to t_0} \dfrac{z(t) - z(t_0)}{t - t_0}$

<u>Baby Lemma</u>: $z'(t)$ exists $\iff$ $x'(t)$, $y'(t)$ exist.

and $z'(t) = x'(t) + iy'(t)$.



Unit complex tangent vector $\dfrac{z'(t_0)}{|z'(t_0)|}$

Def^n: Trace of $\gamma = tr(\gamma) = \{z(t) : a \le t \le b\}$

Chain Rule #1: $g : \Omega_1 \to \Omega_2 \subset \mathbb{C}$ analytic and $f : \Omega_2 \to \mathbb{C}$ analytic, then $h = f \circ g$ is analytic on Ω_1 and $h'(z) = f'(g(z))\, g'(z)$.

Pf: $$\frac{f(w) - f(A)}{w - A} = f'(A) + E(w)$$

where $E(w) \to 0$ as $w \to A$. Define $E(A) = 0$.

$$f(\underset{\substack{\uparrow \\ w = g(z)}}{w}) - f(\underset{\substack{\uparrow \\ A = g(a)}}{A}) = f'(A)(w - A) + E(w)(w - A)$$

$$\frac{f(g(z)) - f(g(a))}{z - a} = f'(g(a))\,\frac{g(z) - g(a)}{z - a} + E(g(z))\,\frac{g(z) - g(a)}{z - a}$$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

$$\longrightarrow \quad = f'(g(a))\cdot g'(a) + \underbrace{E(g(a))}_{=0}\cdot g'(a)$$

Note: g analytic $\Rightarrow$ g continuous. So $g(z) \to g(a)$ as $z \to a$ and $E(g(z)) \longrightarrow E(g(a)) = 0$

Chain Rule #2: f analytic and $z(t) = x(t) + i y(t)$ is diff'ble in real sense, then

$$\frac{d}{dt} f(z(t)) = f'(z(t))\, z'(t)$$

complex der. real var t der.

Pf: Similar to above pf. or

$$f(z(t)) = u(x(t), y(t)) + i\, v(x(t), y(t))$$

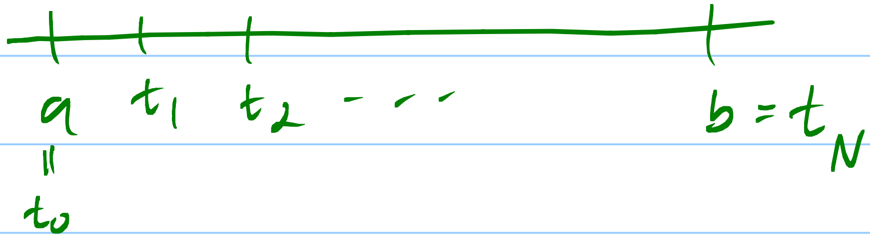
Use $\mathbb{R}^2$ chain rule:

$$\frac{d}{dt} f(z(t)) = (u_x x' + u_y y') + i\,(v_x x' + v_y y')$$

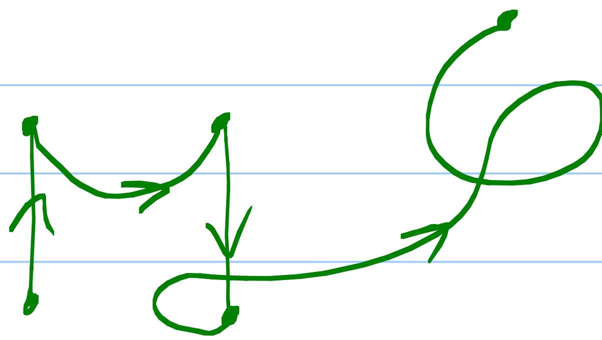
change $\frac{\partial}{\partial y}$'s via C-R Eqns. Get

$$= \underbrace{(u_x + i v_x)}_{f'} \cdot (x' + i y') \quad \checkmark$$

Piecewise $\mathcal{C}^1$-smooth paths:



$\longrightarrow \; z(t) = x(t) + i y(t)$



$z(t)$ is cont.

$z'(t)$ is cont on each (t_n, t_{n+1}) and extends cont. to $[t_n, t_{n+1}]$

Corners, but no jumps.

Def^n: $\displaystyle\int_a^b z(t)\,dt = \int_a^b x(t)\,dt + i\int_a^b y(t)\,dt$

Fund Thm Calculus #1: If $z(t)$ is piecewise $\mathcal{C}^1$-smooth, then

$$\int_a^b z'(t)\,dt = z(b) - z(a)$$

$$\|$$

Pf: $\displaystyle\sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} z'(t)\,dt = \sum (\text{real}) + i\,(\text{imag})$

$$= \sum_{i=1}^{N} \left[z(t_i) - z(t_{i-1}) \right]$$

$= z(t_N) - z(t_0)$ telescoping sum.

$= z(b) - z(a)$.

<u>Lemma</u>: $\left| \int_a^b z(t)\, dt \right| \le \int_a^b |z(t)|\, dt$

<u>Pf</u>; Let $c = \int_a^b z(t)\, dt$. $c = re^{i\theta}$

where $r = \left| \int_a^b z(t)\, dt \right|$ and $\theta \in \arg c$.

Let $\lambda = \frac{\bar{c}}{|c|} = e^{-i\theta}$ ← conjugate of $e^{i\theta}$.

$$\underbrace{\left| \int_a^b z(t)\, dt \right|}_{\text{real}} = \underbrace{\lambda \int_a^b z(t)\, dt}_{= r} \underset{\text{show this.}}{=} \int_a^b \lambda z(t)\, dt$$

$$= \int_a^b Re[\lambda z(t)]\, dt + i \underbrace{\int_a^b Im[\lambda z(t)]\, dt}_{\text{must } = 0}$$

$$= \int_a^b Re[\lambda z(t)]\, dt \le \int_a^b \left| Re[\lambda z(t)] \right| dt$$

$$\le \int_a^b \underbrace{|\lambda z(t)|}_{|\lambda|\,|z(t)|} dt \qquad (\text{because } |Re\, w| \le |w|)$$

$|\lambda| = 1$

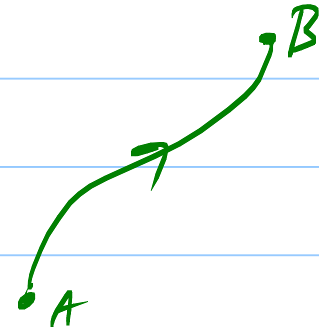
✓

Def^n: Suppose $f : tr(\gamma) \to \mathbb{C}$ is continuous. Then

$$\int_\gamma f\, dz = \int_a^b f(z(t))\, z'(t)\, dt$$

Fund Thm Calc #2: If f is analytic on an open set containing $tr(\gamma)$ and f' is continuous there, then

$$\int_\gamma f'\, dz = f(\underset{\uparrow \atop B}{\text{END}}) - f(\underset{\uparrow \atop A}{\text{START}})$$



Remark: Later, see that f analytic $\Rightarrow$ f' analytic $\Rightarrow$ f'' analytic, so f' will cont. So red stuff will be superfluous.

Pf:

$$\int_\gamma f'\, dz = \int_a^b \underbrace{f'(z(t))\, z'(t)}_{= \frac{d}{dt} f(z(t))}\, dt \qquad \text{Ch. Rule \#2}$$

$$\underset{\uparrow \atop \text{Fund. Thm. Calc \#1}}{=} f(z(b)) - f(z(a)) \qquad \checkmark$$

<u>Basic Estimate:</u> $\left| \int_\gamma f \, dz \right| \leq M \, \text{Length}(\gamma)$

where $M = \underset{a \leq t \leq b}{\text{Max}} |f(z(t))| = \underset{tr(\gamma)}{\text{Max}} |f|$

and $\text{Length}(\gamma) = \int_a^b \underbrace{|z'(t)|}_{\sqrt{x'(t)^2 + y'(t)^2}} \, dt$