Basic Estimate:

$$\left| \int_\gamma f \, dz \right| \le M \cdot \text{Length}(\gamma)$$

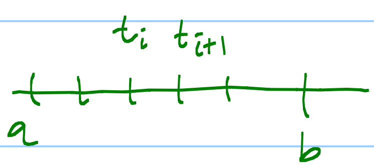
$$M = \sup_\gamma |f| \,, \quad \text{Length}(\gamma) = \int_a^b |z'(t)| \, dt$$

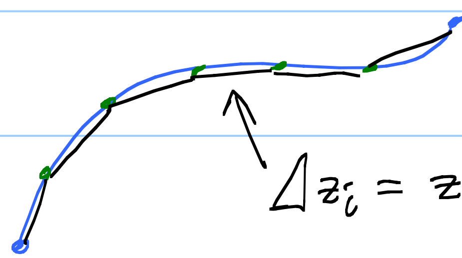
$$(\gamma : z(t),\ a \le t \le b).$$

Pf: $\left| \int_\gamma f \, dz \right| = \left| \int_a^b f(z(t)) \, z'(t) \, dt \right| \le$

$$\int_a^b \underbrace{|f(z(t))|}_{\le M} |z'(t)| \, dt \le M \underbrace{\int_a^b |z'(t)| \, dt}_{\text{Length}} \checkmark$$

Konrad Knopf:





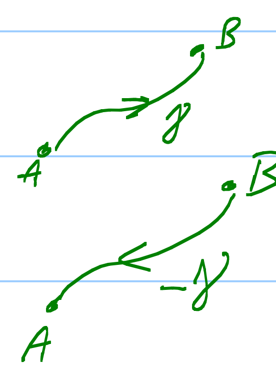
$$\Delta z_i = z(t_{i+1}) - z(t_i)$$

$$\int_\gamma f \, dz = \lim_{|P| \to 0} \sum f(z(t_i)) \, \Delta z_i$$

why: $\Delta z_i \approx z'(t_i) \, \Delta t_i$

Def^n: $\gamma : z(t), \quad a \le t \le b$

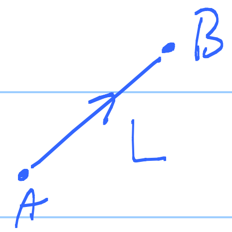
$$-\gamma : z(b + t(a-b)), \quad 0 \le t \le 1$$



Fact: $\int_{-\gamma} f \, dz = - \int_\gamma f \, dz$

Pf: Separate into real and imag parts and use MA 261 change of vars formula.

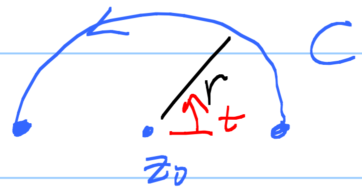
EX:



$L: z(t) = A + t(B-A), \quad 0 \le t \le 1$

$$z'(t) = (B-A)$$

EX:



$C: z(t) = z_0 + \underbrace{re^{it}}_{rE(it)}, \quad 0 \le t \le \pi$

$$z'(t) = 0 + r\underbrace{E'(it)}_{\substack{\shortparallel \\ E(it)}} \underbrace{\frac{d}{dt}(it)}_{i}$$

$$z'(t) = ire^{it}$$

EX: $\frac{d}{dz}(z) = 1$, $\frac{d}{dz}(z^2) = \text{product rule} = 2z$

$$\frac{d}{dz}(z^3) = \frac{d}{dz}(z \cdot z^2) = \text{prod. rule} = 3z^2, \text{ etc.}$$

$$\cdots \text{ induction } \cdots \quad \frac{d}{dz}(z^n) = nz^{n-1}$$

Thm: 1) Complex polys are analytic.

2) Complex polys have complex poly antiderivatives.

Pf: $\frac{d}{dz}\left(\frac{1}{n+1} z^{n+1}\right) = z^n$

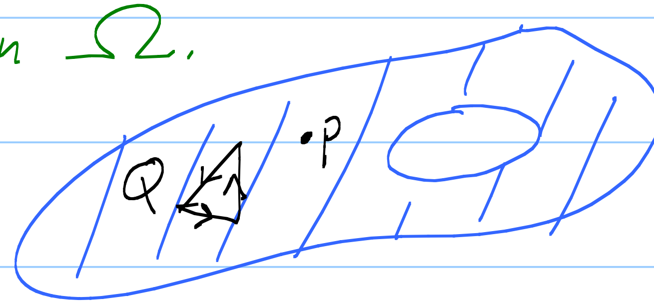
Consequence: $\int_\gamma P(z)\,dz = \int_\gamma Q'(z)\,dz = Q(\text{END}) - Q(\text{START})$

$= 0$ if γ closed.

Note: $\int_\gamma Az+B\,dz = 0$, γ closed.

Goursat's Lemma: $\Omega \subset \mathbb{C}$ open, $p \in \Omega$, f continuous on Ω and analytic on $\Omega - \{p\}$, and Q is a triangle in Ω.

Then $\int_Q f\,dz = 0$.

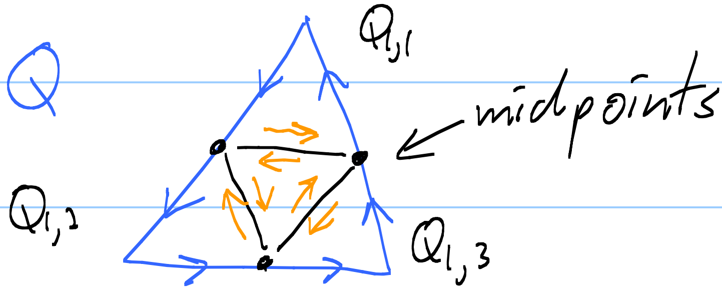


Pf: Case p not in Q. Let $Q_0 = Q$.

Baby Lemma: If $I = a_1 + a_2 + a_3 + a_4$, then $\exists j$ with $|a_j| \geq \frac{|I|}{4}$.

Baby pf: $I=0$ case easy. $I \neq 0$: $|a_j| < \frac{|I|}{4}$, then

$$|I| = \left|\sum a_j\right| \leq \sum |a_j| < |I| \quad ⚡$$



$$\underbrace{\int_Q f\,dz}_{I} = \sum_{j=1}^{4} \int_{Q_{1,j}} f\,dz$$

Get $Q_1 = Q_{1,j_1}$ with $\left|\int_{Q_1} f\,dz\right| \geq \frac{|I|}{4}$.

Repeat: Cut Q_1 into $Q_{2,1}, Q_{2,2}, \ldots, Q_{2,4}$.

Get $Q_2 = Q_{2,j_2}$ with

$$\left|\int_{Q_2} f\,dz\right| \geq \frac{1}{4}\left|\int_{Q_1} f\,dz\right| \geq \frac{|I|}{4^2}$$

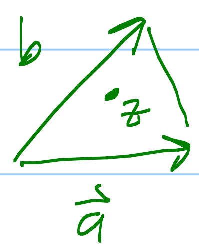
Get $Q_0 \supset Q_1 \supset Q_2 \supset \cdots$

with $\left|\int_{Q_n} f\,dz\right| \geq \dfrac{|I|}{4^n}$.

Topology Lemma: Suppose $K_n \neq \emptyset$ are compact in $\mathbb{R}^2$ and $K_0 \supset K_1 \supset K_2 \supset \cdots$. Then $\bigcap_{n=0}^{\infty} K_n \neq \emptyset$.

Furthermore, if $\operatorname{diam}(K_n) \to 0$ an $n \to \infty$, then $\bigcap_{n=0}^{\infty} K_n = \{z_0\}$, a single pt.

Δ pf:



$z = x\vec{a} + y\vec{b}$ $\quad x, y$ "coords"

Take left most lower most corner of Q_n's. Get non-increasing seq of x_n and y_n bounded below by 0. They converge. Then $z_0 = x_0\vec{a} + y_0\vec{b}$.

Back to pf: $\bigcap Q_n = \{z_0\}$. Let $\varepsilon > 0$.

$$\frac{f(z) - f(z_0)}{z - z_0} = f'(z_0) + E(z) \qquad \Big[\text{Define } E(z_0) = 0$$

$$\boxed{f(z) = f(z_0) + f'(z_0)(z - z_0) + E(z)(z - z_0)}$$

f complex diff'ble at z_0. So $\exists\ \delta > 0$

such that $|E(z)| < \varepsilon$ if $z \in D_\delta(z_0)$.

$\exists N$ such that $Q_n \in D_\delta(z_0)$ for $n \geq N$.

$$\int_{Q_n} f\,dz = \underbrace{\int_{Q_n} (\text{Complex linear})\,dz}_{=0} + \int_{Q_n} E(z)(z-z_0)\,dz$$

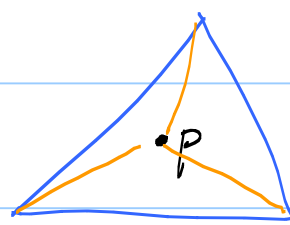
$$\frac{|I|}{4^n} \leq \left|\int_{Q_n} f\,dz\right| \leq \left(\max_{z\in Q_n} |E(z)|\,|z-z_0|\right)\left(\text{Length}(Q_n)\right)$$

$$\leq \varepsilon\, \underbrace{\text{Diam}(Q_n)}_{\frac{1}{2^n}\text{Diam}(Q_0)}\ \underbrace{\text{Length}(Q_n)}_{\frac{1}{2^n}\text{Length}(Q_0)}$$

$$\leq C\frac{\varepsilon}{4^n}, \quad C>0 \text{ const.}$$

So $|I| \leq C\varepsilon$. But ε arbitrary! So $|I| = 0$. ✓

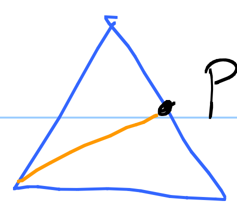
Case p inside Q:



$$\int_Q f\,dz = \sum_{j=1}^{3} \int_{Q_j} f\,dz$$

Reduced to p = vertex case.

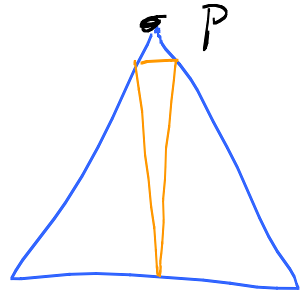
Case p on a side:



$$\int_Q f\,dz = \sum_{j=1}^{2} \int_{Q_j} f\,dz.$$

Reduced to p = vertex case.

Case p = vertex:



$$\int_Q f\,dz = \text{Sum} \int_{Q_j} f\,dz = 0 + 0 + 0 + \int_{Q_{TINY}} f\,dz$$

So
$$\left|\int_Q f\,dz\right| \le \underbrace{\left(\max_{Q_{TINY}} |f|\right)}_{\le M} \underbrace{\text{Length}(Q_{TINY})}$$

$\le M$ (continuous $|f|$ on a closed bounded set)

can make arbitrarily small.

So $\int_Q f\,dz = 0$. Done!