Thm: f analytic on $D_R(z_0)$, $0<r<R$, $a \in D_r(z_0)$.

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{C_r(z_0)} \frac{f(z)}{(z-a)^{n+1}}\, dz$$

Consequently, $f', f'', f''', \ldots$ all analytic!

Pf: Just like HWK 2, Prob. 1. (See Stein)

Start reading Chap. 2.

Cauchy Estimates: Let $z_0 = a$ above, $M = \max_{C_r(z_0)} |f|$.

$$|f^{(n)}(z_0)| \leq \frac{n!\,M}{r^n}$$

Pf: $$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \max\left(\frac{|f(z)|}{r^{n+1}}\right) \underbrace{\text{Length}(C_r)}_{2\pi r}$$

$|z-z_0|=r$

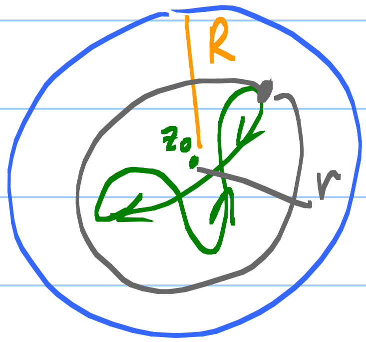
Morera's Theorem: Suppose f is continuous on an open set Ω and $\int_\gamma f\,dz = 0$ for any closed curve in Ω (or even less $\int_Q f\,dz = 0$ for any Δ $Q \subset \Omega$). Then f is analytic on Ω.

Pf: We have Goursat's Lemma as a hyp. Restrict attention to a disc $D_r(z_0) \subset \Omega$, which is convex. Cook up $F(z) = \int_{L_a^z} f\,dw$ as we did for Cauchy Thm.

Then $F' = f$ on Ω, F analytic.
But F analytic $\Rightarrow$ $F' = f$ analytic.

Cor: Power series converge to analytic fcns.

Pf: Let γ be a closed curve inside the disc of convergence. $\gamma : z(t)$, $a \le t \le b$.



Let $r = \max_{a \le t \le b} |z(t) - z_0| < R$

Pick ρ with $r < \rho < R$.

Hadamard $\Rightarrow$ $f_n = \sum_{k=0}^{n} a_k (z - z_0)^k$

Converge uniformly on $\overline{D_\rho(z_0)}$ to $f = \sum_{k=0}^{\infty}$.

Know $\int_\gamma f_n \, dz = 0$.
polynomial

Fact: Uniformly convergent continuous fcns converge to a continuous fcn. So f is continuous.

$$\left| \int_\gamma f \, dz - \underbrace{\int_\gamma f_n \, dz}_{=0} \right| = \left| \int_\gamma (f - f_n) \, dz \right|$$

$$\le \underbrace{\left(\max_{\gamma} |f - f_n| \right)}_{} \text{Length}(\gamma)$$

Can make arbitrarily small by unif conv. on $\overline{D_\rho(z_0)}$.

So $\int_\gamma f\,dz = 0$. Morera's $\Rightarrow$ f analytic.

Def^n: Given a seq. f_n of continuous fcns on an open set Ω, we say that f_n "converges uniformly on compact subsets of Ω" to f if, given a compact set $K \subset \Omega$ and $\varepsilon > 0$, there is an N such that

$$|f(z) - f_n(z)| < \varepsilon \quad \text{for } \underline{\text{all}} \quad z \in K$$

when $n > N$. Notation: Write $f_n \twoheadrightarrow f$.

Exercise: Power series converge unif on compact subsets inside Ω = circle of conv.

Thm: If f_n analytic on Ω and $f_n \twoheadrightarrow f$, then f is analytic!

[Way false $\mathbb{R} \to \mathbb{R}$! Weierstraß:

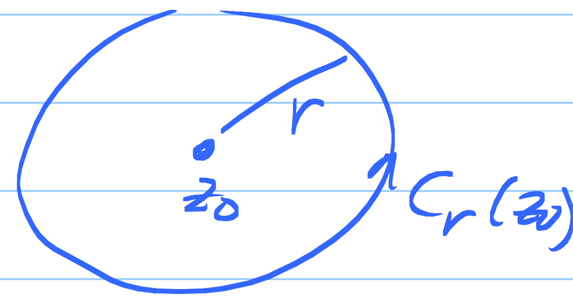
$\sum_{n=1}^{\infty} \frac{1}{2^n} \sin(2^n t)$ converges to a nowhere diff'ble fcn!]

Pf of Thm: Just like power series pf using Morera's and $K = \mathrm{tr}(\gamma)$.

Liouville's Theorem: A bounded entire fcn must be constant.

Words: "entire" means analytic on $\mathbb{C}$.
f "bounded on $\mathbb{C}$" means $\exists\ M>0$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$.

Pf: Pick $z_0 \in \mathbb{C}$.



Cauchy Estimate, $n=1$:

$$|f'(z_0)| \leq \frac{1!\ \max_{C_r(z_0)} |f|}{r^1} \leq \frac{M}{r} \quad \text{if } f \text{ bdd.}$$

Let $r \nearrow \infty$. See $f'(z_0) = 0$. z_0 arbitrary
$\Rightarrow f' \equiv 0 \Rightarrow f$ constant.

Lemma: Basic Polynomial Estimate

Suppose $P(z) = a_N z^N + a_{N-1} z^{N-1} + \cdots + a_1 z + a_0$ is a complex polynomial of degree $N \geq 1$ (i.e., $a_N \neq 0$). Then there are constants

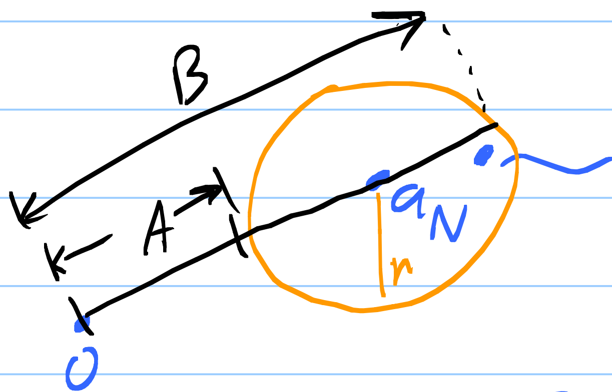
$0 < A < B$ and a radius $R > 0$ so that

$$A|z|^N \le |P(z)| \le B|z|^N \text{ if } |z| > R.$$

<u>Remark</u>: A and B can be taken as close to $|a_N|$ as desired: $0 < A < |a_N| < B$.

<u>Pf</u>: $$\frac{P(z)}{z^N} = a_N + \underbrace{\left[\frac{a_{N-1}}{z} + \cdots + \frac{a_1}{z^{N-1}} + \frac{a_0}{z^N}\right]}_{H(z)}$$

<u>Note</u>: $H(z) \to 0$ as $|z| \to \infty$.



$a_N + H(z)$

$(|H(z)| < r \text{ if } |z| > R)$

Take $r = \frac{|a_N|}{2}$. Then $A = \frac{|a_N|}{2}$, $B = \frac{3|a_N|}{2}$.

<u>Fund. Thm Alg</u>: Pf: Suppose P is a complex poly of deg $N \ge 1$ with <u>no zero</u> in $\mathbb{C}$.

Big Idea: $\frac{1}{P(z)}$ is a <u>bounded</u> entire fcn.

Basic Poly Est $\Rightarrow$ $\left|\frac{1}{P(z)}\right| \le \frac{1}{A|z|^N}$ if

$|z| \geq R$. Get $\frac{1}{P(z)}$ bounded by

$$M = \mathrm{Max}\left(\underline{\max_{|z| \leq R} \left|\frac{1}{P(z)}\right|}, \ \frac{1}{A|z|^N} < \frac{1}{\underline{AR^N}}, \ |z| > R \right)$$

Liouville's $\Rightarrow$ $\frac{1}{P(z)}$ is const. $\Rightarrow$

$P(z) =$ const. Crazy!

$$\frac{d^N}{dz^N}\left[P(z)\right] = N!\, a_N \leftarrow \text{not zero.}$$