Thm: Ω convex open. f analytic on Ω

$\iff \int_\gamma f\,dz = 0$ for all closed γ in Ω.

Warning: Ex: $\Omega = \{z : r < |z| < R\}$.

$\frac{1}{z}$ analytic on Ω, but $\int_{C_\rho} \frac{1}{z}\,dz = 2\pi i$

$r < \rho < R$.

Lemma: $S_N(z) = 1 + z + z^2 + \cdots + z^N$

$$\frac{1}{1-z} = S_N(z) + \underbrace{\frac{z^{N+1}}{1-z}}_{E_N(z)}.$$

If $|z| < \rho < 1$, then $|E_N(z)| < \frac{\rho^{N+1}}{1-\rho}$

Why: $|1-z| \geq \big||1| - |z|\big| = 1 - |z| > 1 - \rho$

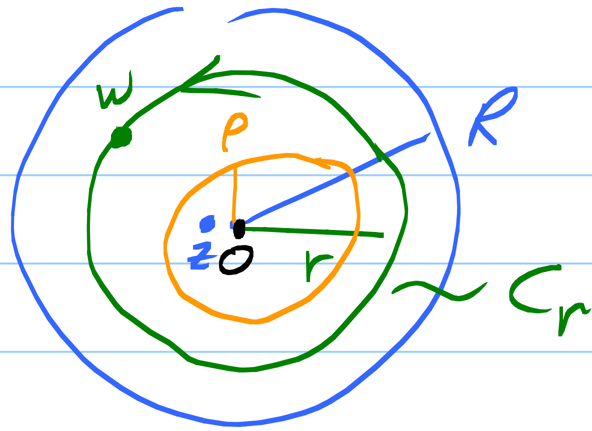
if $|z|<1$ if $|z|<\rho$.

Thm: Suppose f is analytic on $D_R(z_0)$.
Then f is given by a convergent power series $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ where
$a_n = \frac{f^{(n)}(z_0)}{n!}$ and the R. of C. is $\geq R$.

Pf: We may assume that $z_0 = 0$ via a

change of vars $\zeta = z - z_0$. Let $0<\rho<r<R$.

$$\left|\frac{z}{w}\right| = \frac{|z|}{r} < \left(\frac{\rho}{r}\right) < 1$$



$$f(z) = \frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{w}\,\frac{1}{1-\frac{z}{w}}\,dw$$

$$= \frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{w}\left(\left[1+\left(\frac{z}{w}\right)+\cdots+\left(\frac{z}{w}\right)^N\right] + E_N\left(\frac{z}{w}\right)\right)dw$$

$$= \sum_{n=0}^{N} \underbrace{\left(\frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{w^{n+1}}\,dw\right)}_{\frac{f^{(n)}(0)}{n!}} z^n + \underbrace{\frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{w} E_N\left(\frac{z}{w}\right)dw}_{\mathcal{E}_N(z)}$$

Finally, $|\mathcal{E}_N(z)| \le \frac{1}{2\pi}\left(\max_{C_r} \frac{|f(w)|}{r}\cdot\frac{\left(\frac{\rho}{r}\right)^{N+1}}{1-\frac{\rho}{r}}\right)\underbrace{2\pi r}_{\text{length } C_r}$

$$\le \frac{M\left(\frac{\rho}{r}\right)^{N+1}}{1-\frac{\rho}{r}} \longrightarrow 0 \text{ as } N\to\infty.$$

So power series converges uniformly on $D_\rho(0)$.

Can take ρ as close to R as desired,

So R. of. C. is $\geq R$.

Picture:

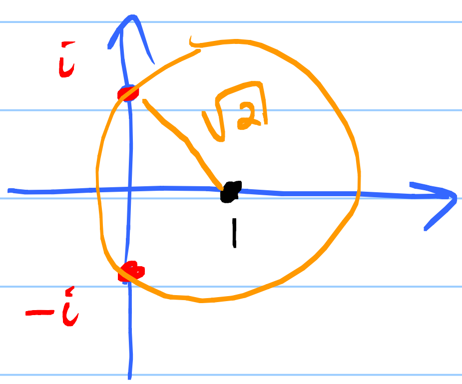
Consequence: $e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$ R. of C. $= \infty$.

Defn: $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

$\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$

$\tan z = \dfrac{\sin z}{\cos z}$

EX: $f(z) = \frac{1}{z^2+1}$, $z_0 = 1$. $f(z) = \sum_{n=0}^{\infty} a_n (z-1)^n$



R. of C. $\geq \sqrt{2}$.

But $R \leq \sqrt{2}$ too.

Because, if $R > \sqrt{2}$, power series converges to $F(z)$ analytic on bigger disc. But $F(z) = f(z)$ on $D_{\sqrt{2}}(1)$. f blows up at $\pm i$, F does not. $\lightning$.

<u>Thm</u>: Suppose $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ with R. of C. $R_f > 0$. We know f is analytic on $D_{R_f}(z_0)$, so f' is analytic there too.

Last theorem $\Rightarrow$ $f' = \sum_{n=0}^{\infty} b_n (z-z_0)^n$ with R. of C. $R_{f'} \geq R_f$. Actually, $R_{f'} = R_f$ and $b_{n-1} = n a_n$, i.e., we can diff'tiate power series term by term.

<u>Pf</u>: $b_{n-1} = \dfrac{(f')^{n-1}(z_0)}{(n-1)!} = \dfrac{n f^{(n)}(z_0)}{n (n-1)!} = n \dfrac{f^{(n)}(z_0)}{n!} = n a_n$

Know $R_{f'} \geq R_f$. If $R_{f'} > R_f$, then power series for f' would converge to G analytic on $D_{R_{f'}}(z_0)$. Let g be an analytic anti-derivative of G. But $g' = f'$ on $D_{R_f}(z_0)$. So $f = g + c$. Oops! This shows power series for f converges on bigger disc. $\lightning$.

<u>Note</u>: Hadamard proof

$$\limsup_{n\to\infty} \sqrt[n-1]{n|a_n|} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

Zeroes of analytic functions: Suppose $f(z)$ is analytic on $D_R(z_0)$, $R>0$, and $f(z_0)=0$. Then $f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n$.

Note: $a_0 = \frac{f^{(0)}(z_0)}{0!} = f(z_0) = 0$.

Dumb case: All a_n's $=0$. Then $f \equiv 0$.

Otherwise, let N be the smallest integer n such $a_n \neq 0$.

Def^n: N is the multiplicity of the zero.

$$f(z) = a_N (z-z_0)^N + a_{N+1}(z-z_0)^{N+1} + \cdots$$

$$= (z-z_0)^N \underbrace{\left[a_N + a_{N+1}(z-z_0) + \cdots \right]}_{H(z)}$$

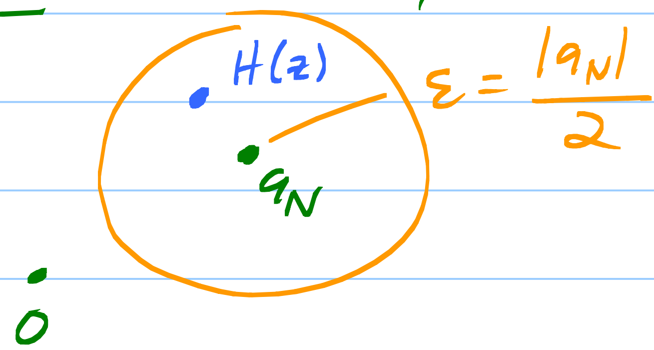
Big point: Power series for H has same R. of C. as power series for f (because they converge for same z).

So H is analytic on $D_R(z_0)$. Note that

$H(z_0) = a_N \neq 0.$

Aha! $f(z) = (z-z_0)^N H(z)$ $\leftarrow$ zeroes cannot pile up at z_0.

<u>Standard analysis</u>: H analytic $\Rightarrow$ H continuous.



$\exists\, \delta > 0$ such that $|H(z) - \underbrace{H(z_0)}_{a_N}| < \varepsilon$ if $|z - z_0| < \delta < R$. So H has <u>no zeroes</u> inside $D_\delta(z_0)$.

Consequentlly, f has only one point (namely $z = z_0$) inside $D_\delta(z_0)$ where it vanishes.

The zero of f at z_0 is <u>isolated</u>.