Theorem: A $\mathcal{C}^2$-smooth harmonic function is *locally* the real part of an analytic function (and is therefore $\mathcal{C}^\infty$-smooth).

Pf: Suppose u is $\mathcal{C}^2$ harmonic on $D_R(z_0)$.

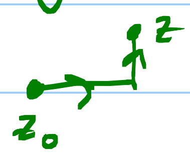
Idea: If we had $f = u + iv$ analytic,

then $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \leftarrow \nabla v = \begin{pmatrix} -u_y \\ u_x \end{pmatrix}$

$$v(z) - v(a) = \int_{\gamma_a^z} \nabla v \cdot d\vec{s} = \int_{\gamma_a^z} \frac{\partial v}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy$$

$$= \int_{\gamma_a^z} -u_y\,dx + u_x\,dy$$

Aha! **Define** $v(z) = \int -u_y\,dx + u_x\,dy$



Step 1: Show that $v_y = u_x$.

$$v = \underbrace{\int_{z_0 \to}}_{\text{const}} + \int_{x+iy_0}^{z = x+iy}$$

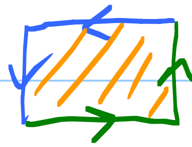
$z_0 = x_0 + iy_0$

$x(t) \equiv x$

$y(t) = t, \quad y_0 \le t \le y$

$$V = \text{const.} + \int_{y_0}^{y} -u_y(x,t)\cdot \overset{\swarrow \frac{dx}{dt}}{0}\cdot dt + \int_{y_0}^{y} u_x(x,t)\cdot \overset{\swarrow \frac{dy}{dt}}{1}\cdot dt$$

Fund. Thm. Calc. $\Rightarrow$ $\frac{\partial v}{\partial y} = u_x(x,y)$ ✓

<u>Step 2</u>: γ  R Green's Thm:

$$\int_\gamma \underset{-u_y}{\underset{\uparrow}{P}}\, dx + \underset{u_x}{\underset{\uparrow}{Q}}\, dy = \iint_R \underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}_{\Delta u \equiv 0} dx\, dy$$

So $v = \int_{\rightarrow\uparrow} = \int_{\uparrow\rightarrow}$

<u>Step 3</u>: Show $v_x = \frac{\partial}{\partial x} \int_{\uparrow\rightarrow} = -u_y$

just like step 1.

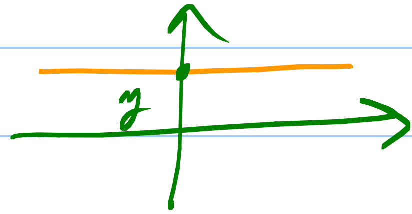
<u>Warning</u>: $\ln|z|$ is harmonic on $D_1(0) - \{0\}$, but it is <u>not</u> globally the real part of an analytic fcn on $\Omega = D_1(0) - \{0\}$. $\leftarrow$ hole!

<u>Prob</u>: Find a harmonic conjugate for

$u = x^2 - y^2 + x + y$

Lemma: $\frac{\partial u}{\partial x} \equiv 0 \iff u = \text{fcn of } y.$

$\frac{\partial u}{\partial y} \equiv 0 \iff u = \text{fcn of } x.$



$\frac{\partial u}{\partial x} \equiv 0 \Rightarrow u$ const on horizontal lines

Cor: $\frac{\partial u_1}{\partial x} = \frac{\partial u_2}{\partial x} \iff u_1 - u_2 = \text{fcn of } y$

etc.

Back to getting v: Need v with $\boxed{u = x^2 - y^2 + x + y}$

$$\begin{cases} \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y - 1 & (A) \\ \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x + 1 & (B) \end{cases}$$

Step 1: Use (A):

"partial integral"

$$V = \int (2y-1)\,dx = 2xy - x + C(y)$$

arb. fcn of y.

Step 2: Use (B):

Need $\frac{\partial}{\partial y}\Big[\underbrace{2xy - x + C(y)}_{V}\Big] \overset{\text{want}}{=} 2x+1$

x's go away! $\rightarrow$ $2x + 0 + c'(y) = 2x+1$

$$\boxed{c'(y) = 1}$$

So $C(y) = y + K$. Done!

$V = 2xy - x + (y+K)$ $\leftarrow$ uniq up to $+K$

EX: Zero set of the harmonic function

$u(x,y) = y$ is $\mathbb{R}$! Not isolated!

No identity thm for harmonic fcns.

Theorem: Suppose u is C^2-smooth harmonic on a domain Ω. If $u \equiv 0$ on $D_\varepsilon(z_0) \subset \Omega$, then $u \equiv 0$ on Ω.

Pf: Important trick: u harmonic $\Rightarrow F = u_x - i u_y$ analytic.

[Use C-R Eqns and mixed partials =, $\Delta = 0$]

Motivation: $f = \underbrace{u + iv}_{\text{locally}}$. $f' = u_x + i v_x$

$= \underbrace{u_x - i u_y}_{\text{global!}}$

Aha! $u \equiv 0$ on $D_\varepsilon(z_0) \Rightarrow$

$F = u_x - i u_y \equiv 0$ on $D_\varepsilon(z_0)$.

Identity Thm $\Rightarrow F \equiv 0$. So $\nabla u \equiv 0$.

So $u \equiv$ const., which must be zero.

[Note: $\int_{\gamma_a^z} \nabla u \, d\vec{s} = u(z) - u(a)$]

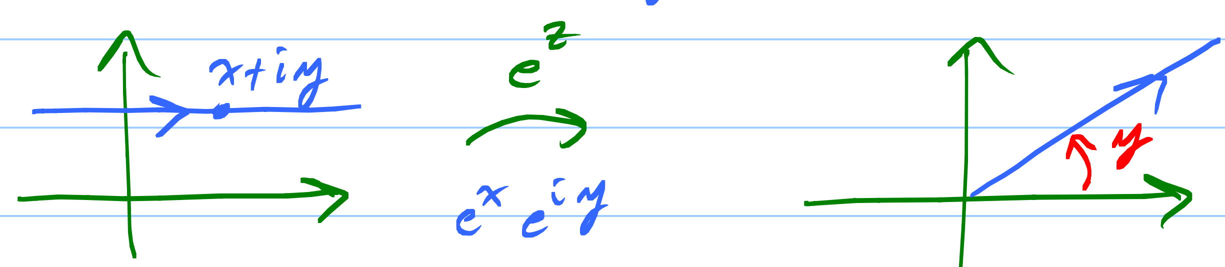
Complex Log fcn: $\log w = \{ z : e^z = w \}$ ← set

$e^z = e^{x+iy} = e^x e^{iy} = w$ ← $e^x = |w|$, $y \in \arg w$

$\log w = \operatorname{Ln}|w| + i \arg w$ ← ∞ many!

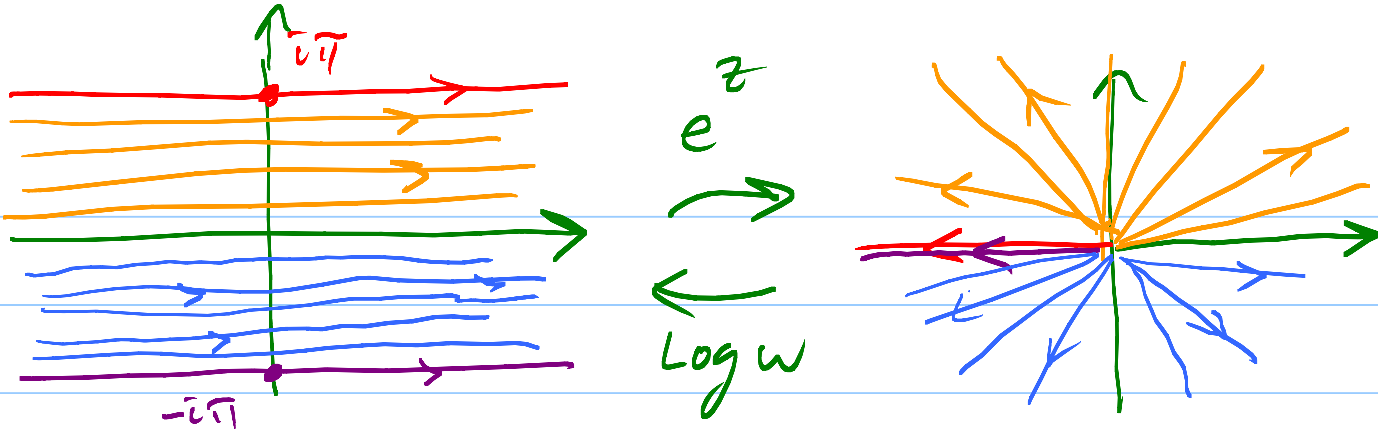
$\operatorname{Log} w = \operatorname{Ln}|w| + i \operatorname{Arg} w$ ← Principal Arg. Principal Branch of log.

$\log w = \{ \operatorname{Ln}|w| + i(\operatorname{Arg} w + 2\pi n) : n \in \mathbb{Z} \}$



$-\infty < x < \infty$

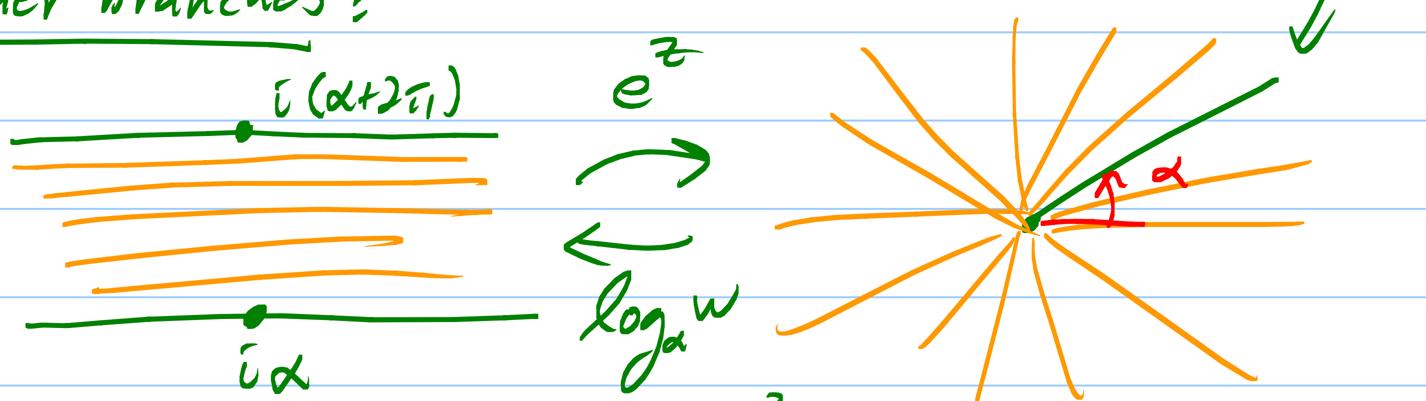
$0 < e^x < \infty$



$\operatorname{Arg} w = \theta, \; -\pi < \theta \leq \pi$, so red wins.

$\mathrm{Log}\, w$ maps $\mathbb{C} - (-\infty, 0]$ one-to-one onto $\{z : -\pi < \operatorname{Im} z < \pi\}$

Other branches:



$$\left(\text{Open horizontal strip}\right) \underset{\log_\alpha w}{\overset{e^z}{\rightleftarrows}} \mathbb{C} - (\text{branch cut})$$

1-1 onto