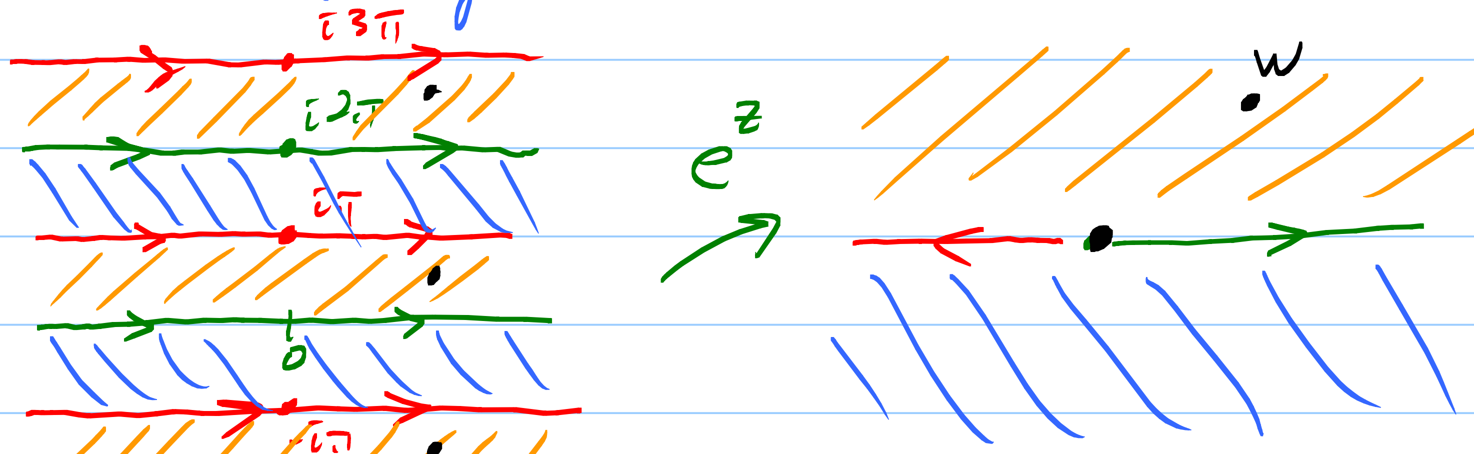
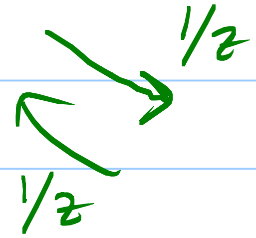
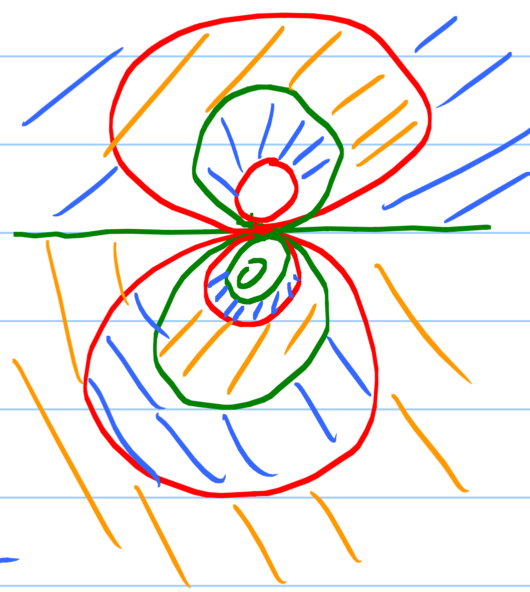
$\log_\alpha z = \operatorname{Ln}|z| + i\theta,\ \theta \in \arg z,\ \alpha < \theta < \alpha + 2\pi$

$\operatorname{Log} z = \operatorname{Ln}|z| + i \operatorname{Arg} z$ ← Principal Arg

$$\log z = \{ w \in \mathbb{C} : e^w = z \}$$
$$= \{ \operatorname{Ln}|z| + i\theta : \theta \in \arg z \}$$
$$= \{ \operatorname{Log} z + in2\pi : n \in \mathbb{Z} \}$$





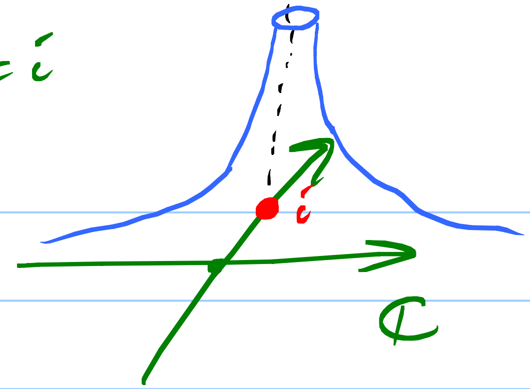


Mind blowing singularity of $e^{1/z}$ at $z=0$. $e^{1/z}$ maps $D_\varepsilon(0) - \{0\}$ <u>onto</u> $\mathbb{C} - \{0\}$ infinite-to-one no matter how small $\varepsilon > 0$ is!

$z=0$ is an "essential singularity" of $e^{1/z}$.

<u>EX</u>: 1) $e^{1/z}$ at $z=0$.

2) $\dfrac{1}{(z-i)^3}$ blows up at $z=i$



$$\left|\frac{1}{(z-i)^3}\right| = \frac{1}{\text{dist}(z,i)^3}$$

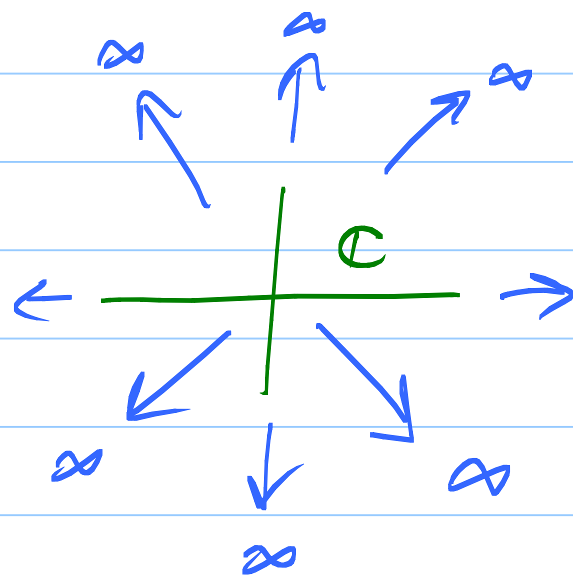
$z=i$ is called a "pole"

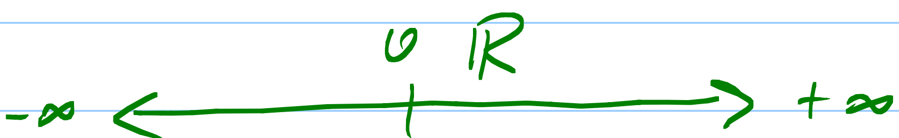
3) $$\frac{\sin z}{z} = \frac{z-\frac{z^3}{3!}+\cdots}{z} = \underbrace{1-\frac{z^2}{3!}+\frac{z^4}{5!}-\frac{z^6}{7!}+\cdots}$$

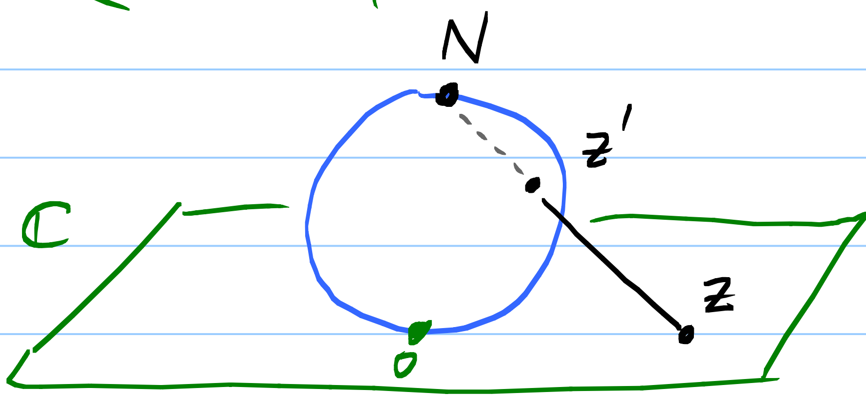
converges for all $z\neq 0$. So $R=\infty$. Entire!

$z=0$ is a "removable singularity"

The "point at infinity":







N corresponds to "the point at ∞". Call it ∞.

$\hat{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$ = extended complex plane.

$D_\varepsilon(\infty) = \{z : |z| > \frac{1}{\varepsilon}\} \cup \{\infty\}$.

Def^n: $\lim_{z \to a} f(z) = \infty$ means, given $M > 0$, there is a $\delta > 0$ such that $|f(z)| > M$ if $|z-a| < \delta$ and $z \neq a$.

Remark: Set $f(a) = \infty$. This makes $f : D_\delta(a) \to \hat{\mathbb{C}}$ continuous at $z=a$.

Riemann Removable Singularity Theorem: Suppose f is analytic on $D_r(a) - \{a\}$, and suppose f is bounded on $D_r(a) - \{a\}$.

Then f can be given a value at $z=a$ to make it analytic on $D_r(a)$.

Way false $\mathbb{R} \to \mathbb{R}$! $h(t) = \sin 1/t$

Pf: Define $H(z) = \begin{cases} (z-a)^2 f(z) & z \neq a \\ 0 & z = a \end{cases}$

Claim: H is analytic on $D_r(a)$.

$$DQ = \frac{H(z) - H(a)}{z-a} = (z-a) f(z) \longrightarrow 0 \text{ as } z \to a.$$

[Let $\varepsilon > 0$. $|f(z)| \le M$ on $D_r(a) - \{a\}$. $\delta = \frac{\varepsilon}{M}$.]

Note: $H(a) = 0$, $H'(a) = 0$.

$$H(z) = \underset{\uparrow \atop 0}{a_0} + \underset{\uparrow \atop 0}{a_1}(z-a) + a_2(z-a)^2 + \cdots$$

$$= (z-a)^2 \Big[\underbrace{a_2 + a_3(z-a) + \cdots}_{h(z) \text{ analytic on } D_r(a)} \Big]$$

$$H(z) = \underbrace{(z-a)^2 h(z)}_{\text{on } D_r(a)} = \underbrace{(z-a)^2 f(z)}_{\text{on } D_r(a) - \{a\}.}$$

So $f(z) = h(z)$ on $D_r(a) - \{a\}$. Let $f(a) = h(a)$.

Alternate Pf:
$$G(z) = \begin{cases} (z-a) f(z), & z \ne a \\ 0, & z = a. \end{cases}$$

G continuous on $D_r(a)$, analytic on $D_r(a) - \{a\}$.

Goursat Lemma, Morera's $\Rightarrow$ G analytic.

$G(a) = 0$. Repeat power series arg.

Def^n: f has an isolated singularity at $z = a$ if there is a $\delta > 0$ so that f is analytic on $D_\delta(a) - \{a\}$.

__Def^n__: 1) a is __removable__ if f can be given a value at a to make f analytic on $D_r(a)$,

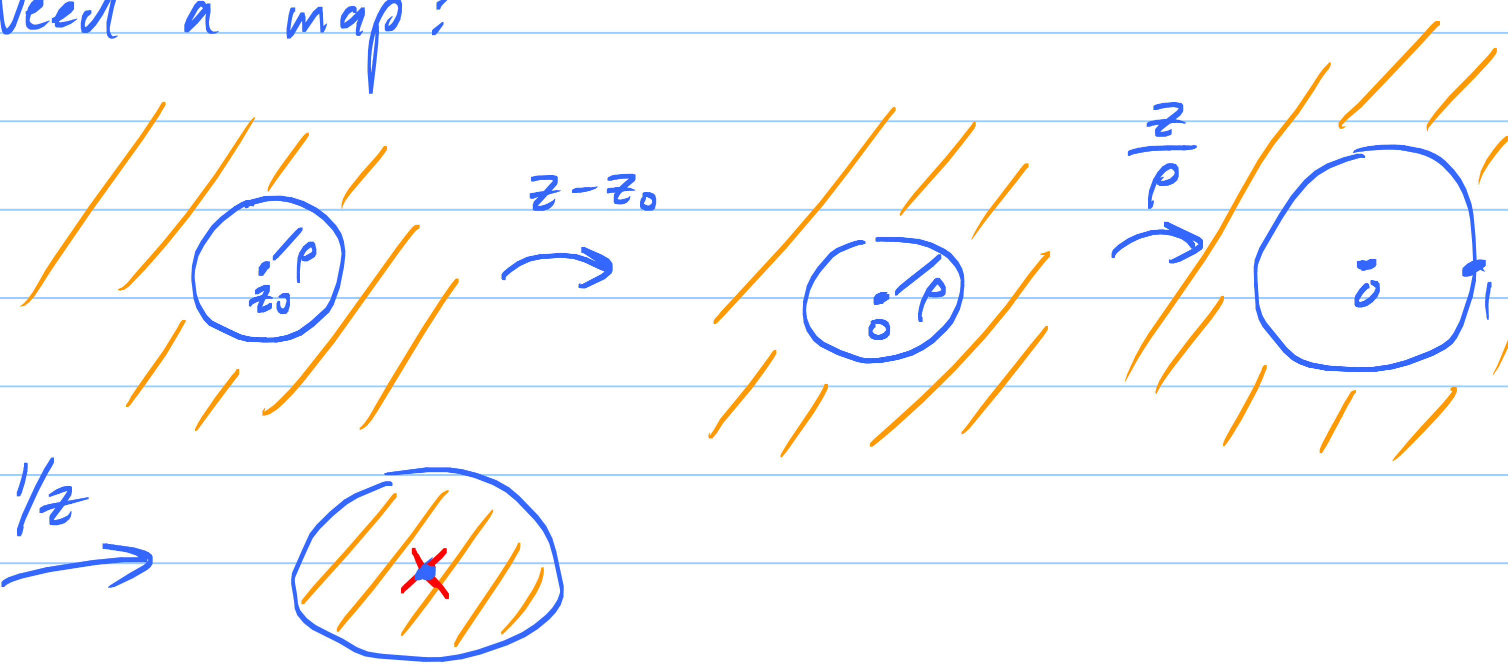
2) a is a __pole__ if $\lim_{z \to a} f(z) = \infty$.

3) a is __essential__ if, for every ε with $0 < \varepsilon < \delta$, $f(D_\varepsilon(a) - \{a\})$ is __dense__ in $\mathbb{C}$.

__Note__: S is dense in $\mathbb{C}$ if every point in $\mathbb{C}$ is a limit point of S.

__Theorem__: 1, 2, 3 are the only possibilities.

Need a map:



$\frac{\rho}{z-z_0}$ maps $\{z : |z-z_0| > \rho\}$ 1-1, onto

$D_1(0) - \{0\}$, analytic.

Prove the Casorati-Weierstraß Thm: An entire function that misses a disc must be constant.