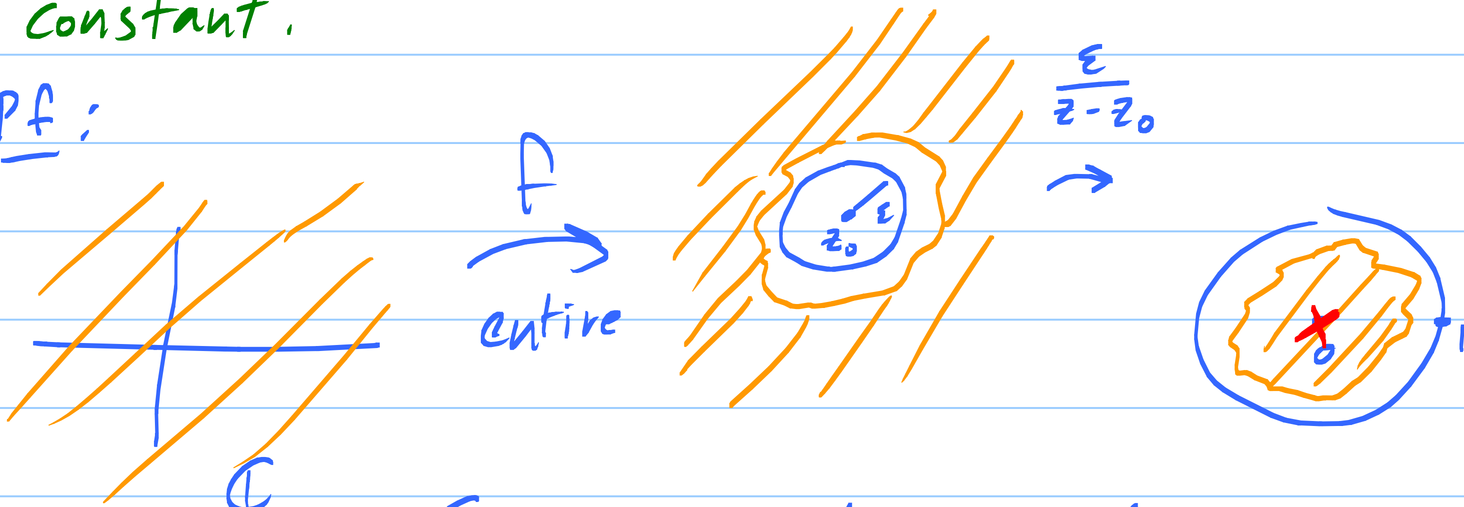
<u>Casorati-Weierstraß Thm</u>: An entire function that misses an open set must be constant.

<u>Pf</u>:



$\frac{\varepsilon}{f(z)-z_0}$ bounded entire! So const.

So f const. ✓

<u>Theorem</u>: Only 3 kinds of isolated sing: 1) removable, 2) pole, 3) essential.

<u>Pf</u>: Suppose f analytic on $D_r(a)-\{a\}$ and a is <u>not</u> essential. Then $\exists\, \varepsilon$ with $0<\varepsilon<r$ such that $f(D_\varepsilon(a)-\{a\})$ is <u>not</u> dense in $\mathbb{C}$. So $\exists\, w_0 \in \mathbb{C}$ which is <u>not</u> a limit pt of this set. So $\exists$ radius $\rho>0$ such that $D_\rho(w_0)\cap f(D_\varepsilon(a)-\{a\})=\emptyset$.

Aha! $h(z) := \dfrac{\rho}{f(z)-w_0}$ is bounded on $D_\varepsilon(a)-\{a\}$.

R.R.S.T. $\Rightarrow$ a removable

Case 1: $h(a) \neq 0$. Then $f(z) = w_0 + \dfrac{\rho}{h(z)}$

see that a is removable for f.

Case 2: $h(a) = 0$. Then $\lim_{z \to a} f(z) = \infty$.

So f has a pole.

Be more detailed: Easy to see $h \not\equiv 0$.

So a is a zero of some finite order

N. $h(z) = (z-a)^N H(z)$ where H analytic

and $H(a) \neq 0$.

Fact: $h(z) = (z-a)^N H(z)$, H analytic, $H(a) \neq 0$

$\Leftrightarrow$ a is a zero of order N.

$$f(z) = w_0 + \frac{\rho}{(z-a)^N H(z)}$$

$$= \frac{1}{(z-a)^N}\Big[\underbrace{\frac{\rho}{H(z)}}_{a_0 + a_1(z-a) + \cdots}\Big] \qquad + \qquad w_0$$

$$= \underbrace{\frac{a_0}{(z-a)^N} + \cdots + \frac{a_{N-1}}{(z-a)}}_{\text{Principal Part}} + \underbrace{a_N + a_{N+1}(z-a) + \cdots}_{+w_0}$$

So R. of C. $\geq \varepsilon$.

Convergent power series on $D_\varepsilon(a)$.

Def^n: f has a pole of order N.

Fact: f has a pole of order N $\iff$ $f(z) = (z-a)^{-N} F(z)$ where F analytic near a and $F(a) \neq 0$. $\iff$ $\frac{1}{f}$ has a zero of order N at a.

Pf: Main ingredient: If f has a pole, $\lim_{z \to a} f = \infty$. So $\exists\, \delta > 0$ such that $|f(z)| > 1$ if $|z-a| < \delta$, $z \neq a$. Now $\frac{1}{f}$ is analytic on $D_\delta(a) - \{a\}$ and $\lim_{z \to a} \frac{1}{f} = 0$. So R.R.S.T. $\implies \frac{1}{f}$ analy.

$\frac{1}{f} = h = (z-a)^N H(z)$. Get

$f(z) = (z-a)^{-N} F(z)$ where $F = \frac{1}{H}$.

Def^n: $R(z) = \dfrac{a_N}{(z-a)^N} + \cdots + \dfrac{a_1}{(z-a)}$

(the Principal Part) is the unique fcn of this form such that $f-R$ has a removable sing at a.

Picard's Little Theorem: Suppose f is entire. Then either f is a polynomial, or f assumes every value in $\mathbb{C}$ infinitely many times, with one possible exception.

EX: e^z misses 0. $\sin z$ misses nothing.

Picard's Big Theorem: If f has an essential singularity at a, then on any $D_\varepsilon(a)-\{a\}$ where f is analytic, f assumes every value in $\mathbb{C}$ infinitely many times, with one possible exception.

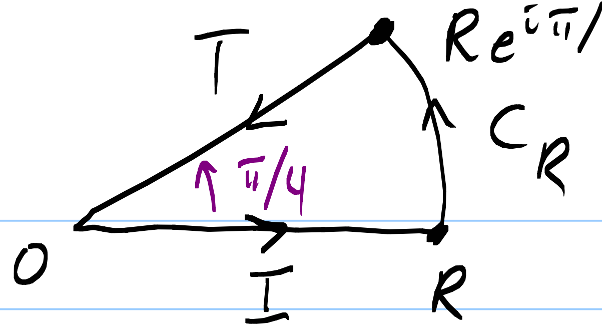
EX: $e^{1/z}$ misses 0. $\sin\frac{1}{z}$ doesn't miss anything.

HWK hints: $|f^{(n)}(a)| > n!\, n^n$ ↯

$n^{n/1000}$?

$(\ln n)^n$?

HWK 4, #10.



$f(z) = e^{-z^2}$

$I: z(t) = t, \quad 0 \le t \le R$

$$\int_I f(z)\,dz = \int_0^R e^{-t^2} \cdot 1\,dt \xrightarrow[R\to\infty]{} \text{known.}$$

$-T: \; z(t) = te^{i\pi/4}, \quad 0 \le t \le R$

$$\int_T f\,dz = -\int_{-T} f\,dz = \text{something you want.}$$

Fact: $\int_{C_R} f\,dz \to 0$ as $R \to \infty$.

$C_R: \; z(t) = Re^{it}, \quad 0 \le t \le \pi/4$

$$\left|\int_{C_R} e^{-z^2}\,dz\right| \le \left(\max_{0\le t\le \frac{\pi}{4}} \left|e^{-R^2e^{i2t}}\right|\right)\left(\frac{\pi}{4}R\right)$$

$$\left|e^{-R^2e^{i2t}}\right| = \left|e^{-R^2\cos 2t}\, e^{-iR^2\sin 2t}\right|$$

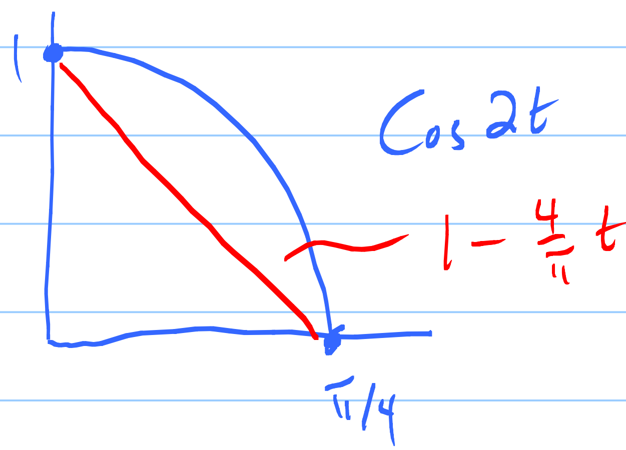
$$= e^{-R^2\cos 2t} \quad \text{has max } 1.$$

Fancier thing: (Jordan's Lemma).

$$\left| \int_{C_R} e^{-z^2}\, dz \right| = \left| \int_0^{\pi/4} e^{-R^2 \cos 2t} e^{-iR^2 \sin 2t}\, iRe^{it}\, dt \right|$$

$$\leq R \int_0^{\pi/4} e^{-R^2 \cos 2t}\, dt$$

Hint:



$$\left(1 - \frac{4}{\pi} t\right) \leq \cos 2t$$

$$-R^2 \cos 2t \leq -R^2\left(1 - \frac{4}{\pi} t\right)$$

$$e^{-R^2 \cos 2t} \leq e^{-R^2\left(1 - \frac{4}{\pi} t\right)}$$