Partial Fractions: Suppose $P(z)$ and $Q(z)$ are polynomials with no common factors and with $N_Q = \deg Q > \deg P = N_P$.

Step 1: Let a_k, $k = 1, \dots, M \le N_Q$ denote the zeroes of Q. Near a_k, $Q(z) = (z-a_k)^{m_k} q_k(z)$ where q_k is a poly with $q_k(a_k) \neq 0$.

Step 2:
$$\frac{P(z)}{Q(z)} = \frac{1}{(z-a_k)^{m_k}} \cdot \frac{P(z)}{q_k(z)}$$

See P/Q has a pole of order m_k at a_k. Let R_k be the principal part at a_k. Know $\frac{P}{Q} - R_k$ has a removable sing at a_k.

Step 3: Let $f = \frac{P}{Q} - \sum_{k=1}^{M} R_k$.

Near a_p:
$$f = \underbrace{\frac{P}{Q} - R_p}_{\text{removable at } a_p} - \underbrace{\sum_{k \neq p} R_k}_{\text{analytic near } a_p}$$

f has removable sing at all a_k's. It is entire!

Claim: $\lim_{z\to\infty} f(z) = 0$. $\leftarrow \lim_{|z|\to\infty} |f(z)| = 0$

Note that $R_k(z) \to 0$ as $z \to \infty$.

Also $\quad A_p |z|^{N_P} \le |P(z)| \le B_p |z|^{N_P}$, $|z| > R_p$

$\quad A_Q |z|^{N_Q} \le |Q(z)| \le B_Q |z|^{N_Q}$, $|z| > R_Q$

$$\left|\frac{P(z)}{Q(z)}\right| \le \frac{B_p}{A_Q} \frac{1}{|z|^{\underbrace{N_Q - N_P}_{\ge 1}}} \quad \text{if } |z| > \max(R_P, R_Q)$$

So $f(z) \to 0$ as $z \to \infty$. So f is bdd entire. Liouville's $\Rightarrow$ $f \equiv$ const, which must be zero. Conclude that

$$\frac{P}{Q} = \sum_{k=1}^{m} R_k \leftarrow \text{Partial Fractions Decomp!}$$

Freshman Calc: $\dfrac{x^3 + 7x + 13}{(x-2)^2(x^2+x+1)} = \dfrac{A}{(x-2)^2} + \dfrac{B}{(x-2)}$

$$+ \frac{Cx + D}{x^2 + x + 1} \leftarrow \frac{E}{(x-r)} + \frac{E}{(x-\bar{r})}$$

Schwarz Lemma (Chapter 8): Suppose $f: D_1(0) \to D_1(0)$ is analytic (i.e., $|f(z)| < 1$ when $|z| < 1$, f analytic), and $f(0) = 0$. Then 1) $|f(z)| \le |z|$ for all z, and 2) $|f'(0)| \le 1$.

Furthermore, if $=$ happens in (1) for some $z \ne 0$, or if $=$ happens in (2), then $f(z) = \lambda z$ for a constant λ with $|\lambda| = 1$, i.e, $\lambda = e^{i\theta}$, f is a rotation.

Pf: Define
$$F(z) = \begin{cases} \dfrac{f(z)}{z} & z \ne 0 \\ f'(0) & z = 0 \end{cases}$$

Note $DQ = \dfrac{f(z) - \overset{0}{\overset{\shortparallel}{f(0)}}}{z - 0} = \dfrac{f(z)}{z}$, so F is cont. at $z = 0$. R.R.S.T. $\Rightarrow$ F analytic on $D_1(0)$

Pick $z_0 \in D_1(0)$ and a radius r with $|z_0| < r < 1$. [Think $r \nearrow 1$]. Max Princ.

$$|F(z_0)| \le \max_{|z|=r} |F| = \max_{|z|=r} \frac{|f(z)|}{|z|} < \frac{1}{r}$$

Now let $r \nearrow 1$. See $|F(z_0)| \le 1$.

z_0 arbitrary, so ineq (1) & (2) hold.

If $|F(z_0)| = 1$ for any z_0, this gives the "furthermore" condition. Max Princ

$\Longrightarrow F(z) \equiv \text{const.}, \; |\underbrace{\text{const}}_{\lambda}| = 1$. So

$F(z) = \frac{f(z)}{z} \equiv \lambda$ if $z \neq 0$. So

$f(z) = \lambda z$ if $z \neq 0$. Holds at $z = 0$ too by continuity.

Logs and roots on a convex open set:

Thm: Given a non-vanishing analytic fcn F on a convex open set Ω, there is an "analytic log" fcn G on Ω for F such that $e^G = F$ on Ω.

Cor: There is an analytic N-th root of F on Ω: $e^{G/N}$.

Idea: Suppose we have G: $\quad F = e^G$

$F' = e^G \cdot G'$

See $G' = \frac{F'}{F}$

Pf of Thm: Define $G(z) = \int_{L_a^z} \frac{F'(w)}{F(w)} dw$. ← fixed

We know $G'(z) = \frac{F'(z)}{F(z)}$. $(*)$

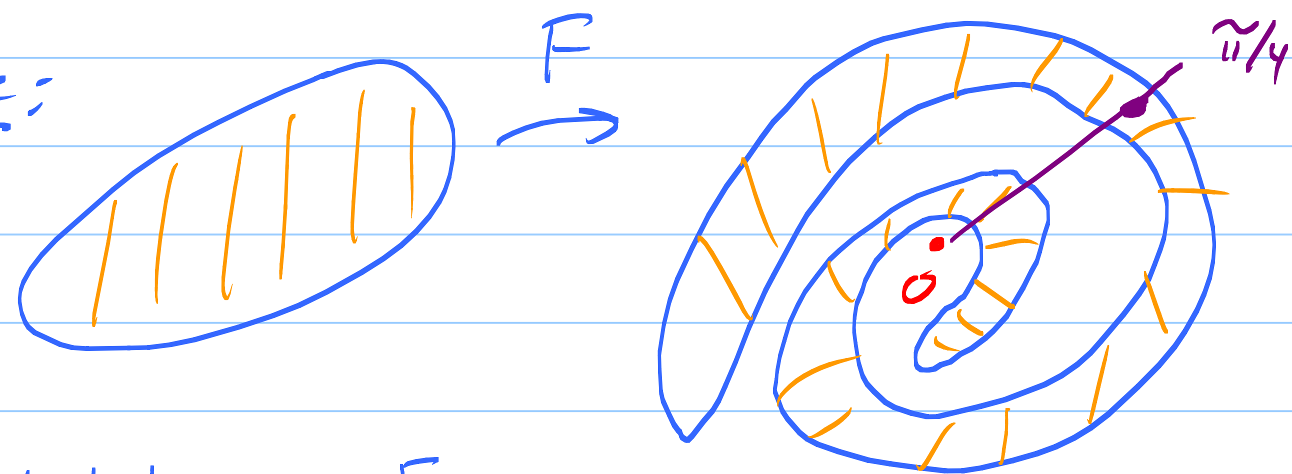
Consider $\frac{F}{e^G} = Fe^{-G}$. Note that

$$\frac{d}{dz}(Fe^{-G}) = (\underbrace{F' - FG'}_{=0 \text{ from } (*)}) e^{-G} \equiv 0$$

So $Fe^{-G} = C$, a const on Ω.

$$F = Ce^G = e^{\operatorname{Log} C} e^G$$
$$= e^{(\operatorname{Log} C + G)}$$ ← the right analytic G

Picture:



$G = \ln|F| + i \arg F$ ← $\arg F$ can be defined in a continuous manner!

<u>Exercise</u>; Suppose H is an N-th root of F on Ω. If $\tilde{H}$ is another analytic N-th root of F on Ω, then show that $\tilde{H} \equiv \lambda H$ where $|\lambda| = 1$, $\lambda^N = 1$.

Hmmm. $N=2$ case: $\tilde{H}^2 = H^2$.

$$\tilde{H}^2 - H^2 = 0$$

$$(\tilde{H} - H)(\tilde{H} + H) = 0 \implies$$ one factor $\equiv 0$ by Identity Thm.