Open Mapping Theorem (O.M.T.):

Non-constant analytic functions map open sets to open sets.

Notation: $f(U) = \{ f(z) : z \in U \}$

$f : \Omega \to \mathbb{C}$. $f^{-1}(U) = \{ z \in \Omega : f(z) \in U \}$

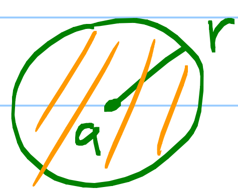
Fact: f continuous $\iff$ $f^{-1}(U)$ open whenever U is open.

Cor of OMT: If analytic f has an inverse, the inverse is continuous.

OMT false $\mathbb{R} \to \mathbb{R}$: $h(t) = 1 - t^2$

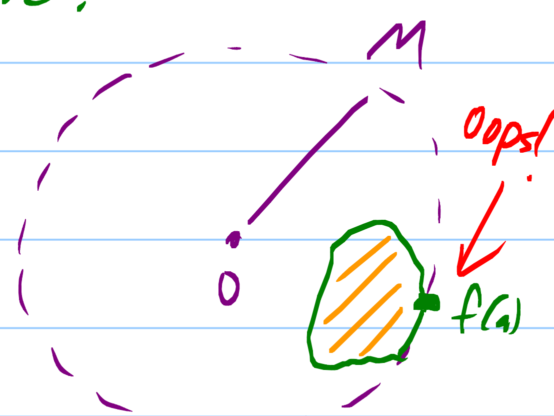
$h : (-1, 1) \to (0, 1]$

Remark: OMT $\Rightarrow$ Max Princ.



$|f(a)| = M$,

max on $\overline{D_r(a)}$



Argument Principle for a disc: Suppose f has a zero of order N at a (f analytic). Then $f(z) = (z-a)^N F(z)$ where F analytic and $F(a) \neq 0$.

$$\frac{f'(z)}{f(z)} = \frac{N(z-a)^{N-1}F(z) + (z-a)^N F(z)}{(z-a)^N F(z)}$$

$$= \underbrace{\frac{N}{(z-a)}}_{\text{princ part}} \overset{\leftarrow \text{ residue}}{} + \frac{F'(z)}{F(z)}$$

$N = \mathrm{Res}_a \frac{f'}{f}$. See $\frac{f'}{f}$ has a pole of order 1 at a.

<u>Argument Principle #1</u>: Suppose f analytic on $D_R(z_0)$ and $0<r<R$. If has no zeroes on $C_r(z_0)$, then

$$\frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f'}{f}\,dz = \begin{pmatrix} \#\text{ zeroes of } f \\ \text{inside } C_r(z_0) \\ \text{counted with mult.} \end{pmatrix}$$

<u>Pf</u>: Let $a_1, \dots, a_n$ denote zeroes of f inside $C_r(z_0)$ and say mult of zero at a_k is m_k. Now $\frac{f'}{f} - \sum_{k=1}^{n} \frac{m_k}{z-a_k} = h$ has removable sing at each a_k. Note that h is analytic on $D_\rho(z_0)$, $\rho > r$. Cauchy's Thm yields

$$\int_{C_r(z_0)} h\,dz = 0$$

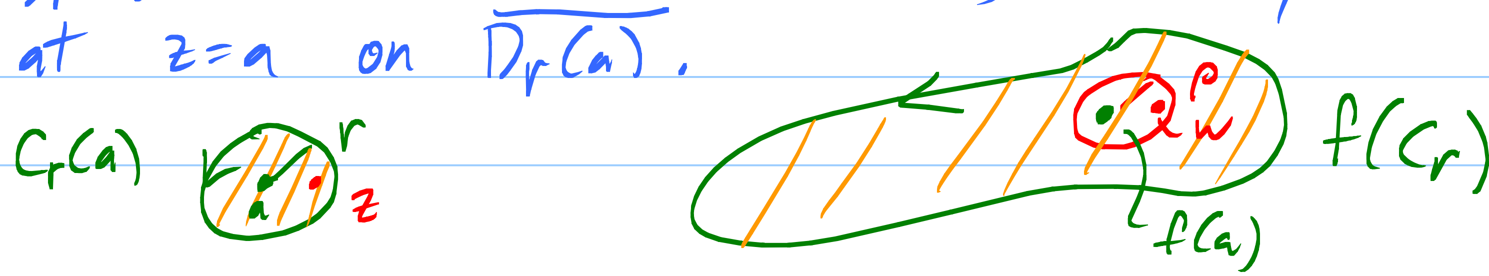
$$\int_{C_r(z_0)} \frac{f'}{f}\,dz - \sum_{k=1}^{n} \underbrace{\int_{C_r(z_0)} \frac{m_k}{z-a_k}\,dz}_{2\pi i\, m_k} = 0 \quad \checkmark$$

<u>Argument Principle #2</u>: Same hyp as #1, but let f have finitely many <u>poles</u> inside $C_r(z_0)$. Replace N by $-N$ in calculation of residue to see $\operatorname{Res}_a \frac{f'}{f} = -(\text{order of pole})$ at a pole a. Same proof yields

$$\frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f'}{f}\,dz = \left(\begin{matrix}\#\text{ zeroes of } f\\ \text{in } C_r(z_0)\end{matrix}\right) - \left(\begin{matrix}\#\text{ poles } f\\ \text{in } C_r(z_0)\end{matrix}\right)$$

counted with mult.

<u>Pf of OMT</u>: Suppose f analytic, non-constant on a domain Ω. Pick $a \in \Omega$. Since f not const, $f(z) - f(a)$ not const, so $f(z) - f(a)$ has an <u>isolated</u> zero at a, <u>say</u> of order N_0. So $\exists\, r > 0$ such that $\overline{D_r(a)} \subset \Omega$ <u>and</u> $f(z) - f(a)$ is only zero at $z = a$ on $\overline{D_r(a)}$.



$f(a)$ not in compact set $f(C_r(a))$.

So $m = \min |f(a) - f(z(t))| > 0$.

Pick ρ with $0 < \rho < m$.

Claim: every $w \in D_\rho(f(a))$ gets hit.

Let $F(z) = f(z) - w$ ← zero means $f(z) = w$.

$F'(z) = f'(z)$

Arg. Princ $\Rightarrow \underbrace{\frac{1}{2\pi i} \int_{C_r} \frac{f'(z)}{f(z)-w}\, dz}_{H(w)} = N$

↑ integer, # zeroes $f(z) - w$

Note: $H(f(a)) = N_0$. ← order of zero of $f(z) - f(a)$ at a.

Claim: $H(w)$ is analytic!

$$H(w) = \frac{1}{2\pi i} \int_a^b \frac{f'(z(t))\, z'(t)}{f(z(t)) - w}\, dt$$

$\frac{d}{dt} f(z(t))$

$$= \frac{1}{2\pi i} \int_{f(C_r(a))} \frac{1}{\zeta - w}\, d\zeta$$

← HWK? #1! Diff'ble.

Note: $C_r(a)$: $z(t)$, $a \le t \le b$

$f(C_r(a)) : f(z(t)),\ a \le t \le b.$

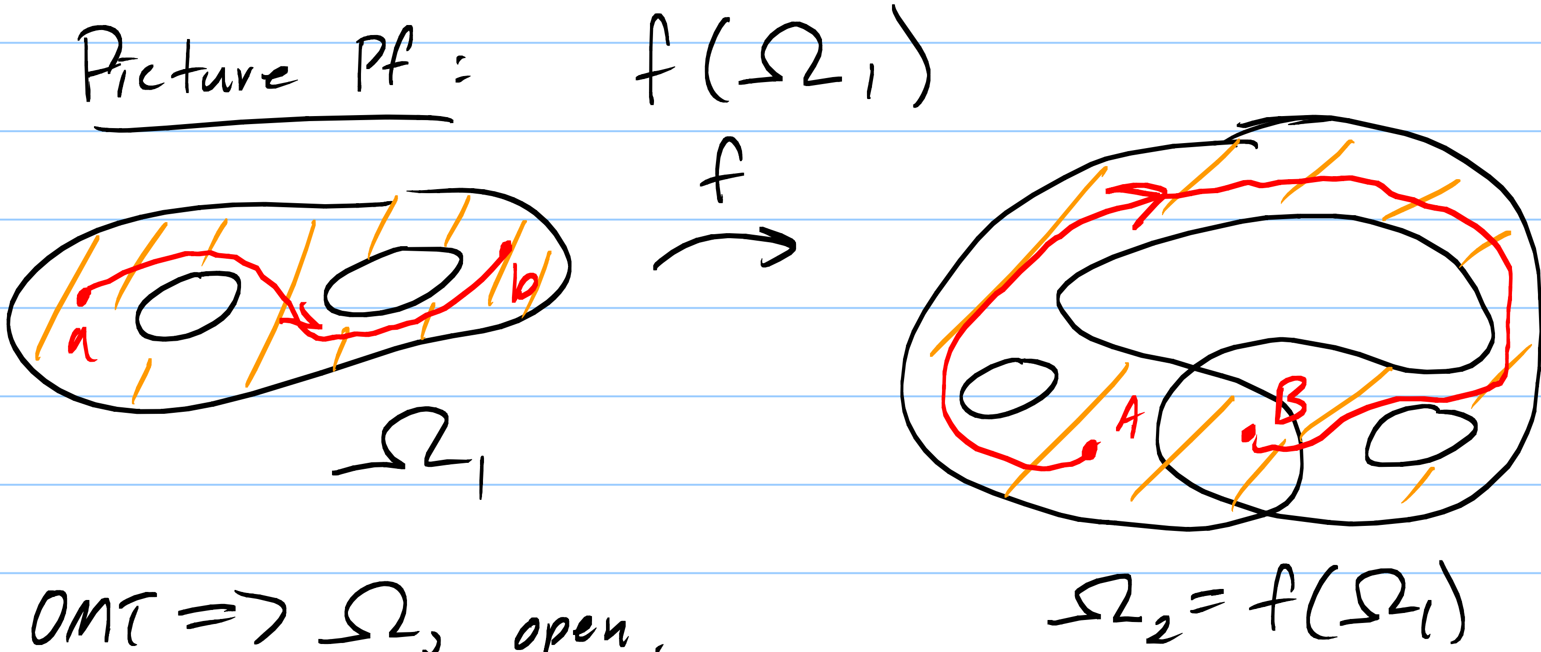
Fact: integer valued analytic fcns on a domain are constant!

So $H(w) \equiv N_0$ on $D_r(a)$. $N_0 \ge 1$. So every w gets hit, i.e., $D_\rho(f(a)) \subset f(D_r(a))$.

Since a arbitrary, see $f(\mathcal{O})$ is open if $\mathcal{O}$ is open.

Cor: Non-constant analytic fcn maps a domain to a domain.

Picture Pf:



OMT $\Rightarrow \Omega_2$ open.

Path connected easy.

Inverse Function Theorem is a corollary of the proof of the OMT.