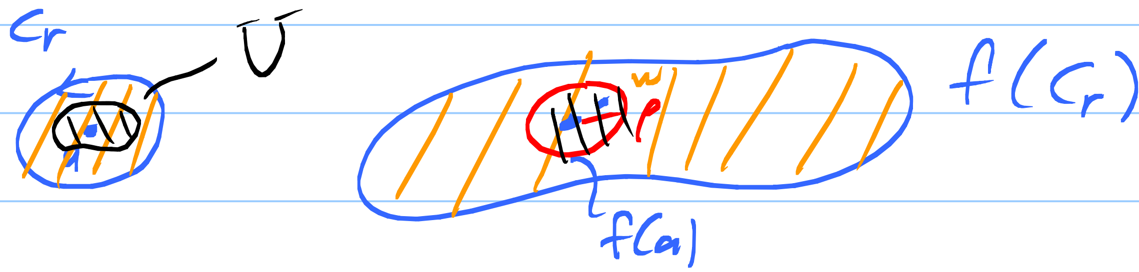
Inverse Function Theorem: Suppose f is analytic on $D_r(a)$ and $f'(a) \neq 0$. Then $\exists\, \rho > 0$ and subdomain U of $D_r(a)$, $a \in U$, such that $f: U \to D_\rho(f(a))$ is one-to-one and onto and f^{-1} is analytic on $D_\rho(f(a))$.

Pf: Go back to proof of O.M.T.



Can shrink r so that f' non-vanishing on $\overline{D_r(a)}$ and a is only zero of $f(z) - f(a)$. Because $f'(a) \neq 0$, zero of $f(z) - f(a)$ at $z = a$ has order one. So $N_0 = 1$. All w's in $D_\rho(f(a))$ get hit exactly once. OMT $\Rightarrow f^{-1}$ cont on $D_\rho(f(a))$.

$$DQ = \frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} \qquad \begin{cases} w = f(z) \\ w_0 = f(z_0) \end{cases} \quad z_0 = a$$

$$= \frac{z - z_0}{f(z) - f(z_0)} \qquad \begin{aligned} z &= f^{-1}(w) \\ z_0 &= f^{-1}(w_0) \end{aligned}$$

$$\to \frac{1}{f'(z_0)} \quad \text{because} \quad z \to z_0 \text{ as } w \to w_0$$

by continuity of f^{-1}

Repeat for any $z_0 \in U$.

Super Inverse Function Theorem: Suppose f is analytic and one-to-one on a domain Ω. Then f' must be non-vanishing on Ω. Consequently (via OMT), $f(\Omega)$ is a domain and $f^{-1}: f(\Omega) \to \Omega$ is analytic (via IFT).

Pf: Case $f' \equiv c$. Then $f(z) = cz + b$, $c \neq 0$.
True and easy.

Otherwise, zeroes of f' are isolated.
Suppose $f'(a) = 0$. Can restrict to $D_r(a)$ with $\overline{D_r(a)} \subset \Omega$ and a is only zero of f' on $\overline{D_r(a)}$. Let $A = f(a)$. OMT gives $f: D_r(a) \xrightarrow[\text{onto}]{1\text{-}1} U$, U domain, and f^{-1} cont on U. I.F.T. $\Rightarrow$ f^{-1} analytic on $U - \{A\}$.
R.R.S.T $\Rightarrow$ f^{-1} analytic on U. Write $F = f^{-1}$.

$$F(f(z)) = z$$

Chain rule $\quad F'(f(z))\, f'(z) = 1$

at $z=a$: $\quad F'(f(a))\, \underbrace{f'(a)}_{\text{can't be zero!}} = 1$

Way false $\mathbb{R} \to \mathbb{R}$. $h(t) = t^3$
$h: (-1,1) \longrightarrow (-1,1)$ one-to-one onto.
h^{-1} not diff'ble at $t = 0$. $h'(0) = 0$.

Remark: $\mathbb{R}^2 \to \mathbb{R}^2$ $\quad f = u + iv$

Jacobian Matrix

$$J = \begin{bmatrix} u_x & \overset{-v_x}{\overset{\shortparallel}{u_y}} \\ v_x & \underset{\underset{u_x}{\shortparallel}}{v_y} \end{bmatrix} = \begin{bmatrix} A & B \\ -B & A \end{bmatrix}$$

$$\det J = u_x^2 + v_x^2 = |f'|^2$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$J^{-1} = \frac{1}{\det} \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$$

See C-R Equns for f^{-1}

Cor of Super Inverse Function Theorem:

$\mathrm{Aut}(\Omega) = \{ f$ analytic on Ω, $f : \Omega \to \Omega$ 1-1, onto $\}$

Automorphism Group Ω.

SIFT $\Rightarrow f^{-1} \in \mathrm{Aut}(\Omega)$ if f is.

Schwarz Lemma + Cor allows us to find $\mathrm{Aut}(D_1(0))$.

Def^n: $\varphi_a(z) = \dfrac{z - a}{1 - \bar{a} z}$, $\quad a \in D_1(0)$,

is a Möbius Transformation.

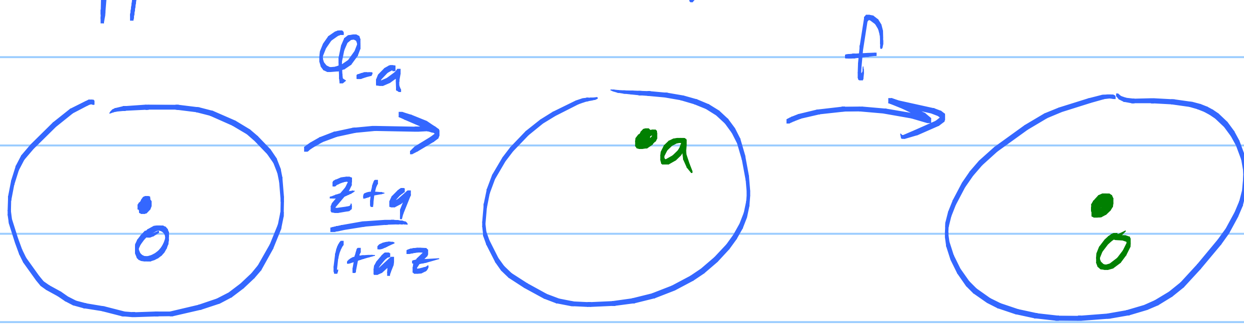
<u>Facts</u>: 1) $\varphi_a : D_1(0) \to D_1(0)$ 1-1, onto analytic.

2) $\varphi_a^{-1} = \varphi_{-a}$ (HWK 1 problems).

$w = \dfrac{z-a}{1-\bar{a}z}$ solve for z.

<u>Thm</u>: $\mathrm{Aut}(D_1(0)) = \{\lambda \varphi_a : |\lambda| = 1,\ a \in D_1(0)\}$

<u>Pf</u>: Suppose $f \in \mathrm{Aut}(D_1(0))$



$a = f^{-1}(0)$

Now $F(z) = f(\varphi_{-a}(z)) : D_1(0) \circlearrowleft$ 1-1 onto

$F(0) = 0$.

Schwarz $\Rightarrow |F'(0)| \le 1$.

But F^{-1} satisfies same hyp. So

$$\underbrace{|(F^{-1})'(0)|}_{\left|\frac{1}{F'(0)}\right|} \le 1.$$

Conclude that $|F'(0)| = 1$.

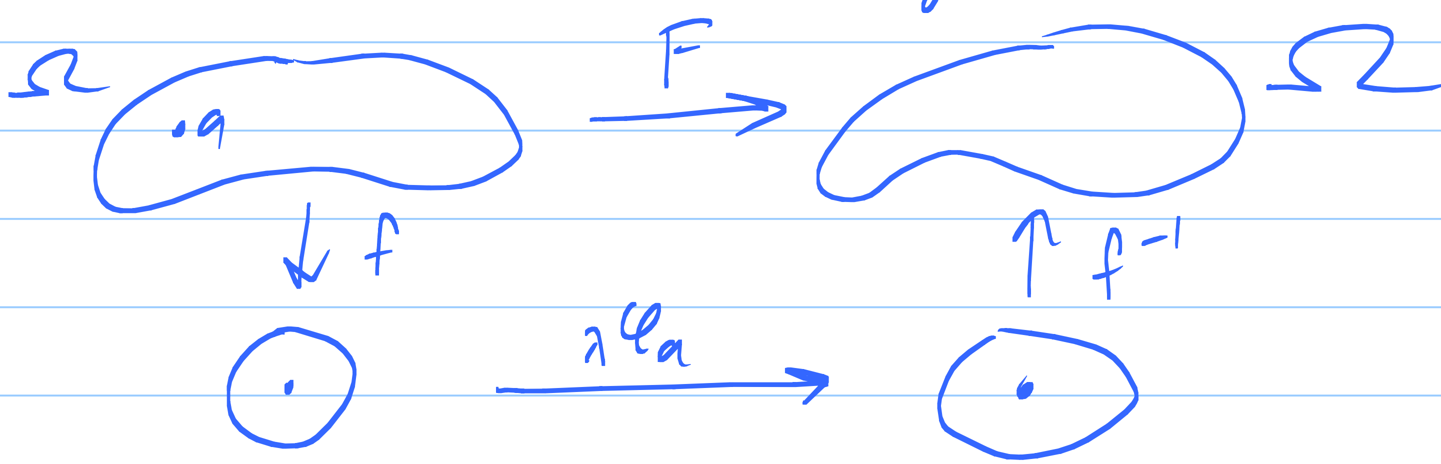
Schwarz Part 2: $F(z) = \lambda z$, $|\lambda| = 1$.

$$F(z) = f(\underbrace{\varphi_{-a}(z)}_{w}) = \lambda \underset{\uparrow\ \varphi_a(w)}{z}$$

<u>Note</u>: $(f^{-1})'(w) = \dfrac{1}{f'(f^{-1}(w))}$ IFT + Chain Rule.

$$(f^{-1})'(f(z)) = \frac{1}{f'(z)}$$

<u>Remark</u>: Riemann Mapping Thm



$Aut(\Omega)$, Ω simply connected domain

$\approx Aut(D, (0))$.

<u>Local Behavior of Analytic Functions</u>:

Suppose f analytic and not const, $f(z_0) = w_0$. Then $f(z) - w_0$ has an isolated zero at z_0, say of order N.

$$f(z) - w_0 = (z - z_0)^N H(z), \quad H(z_0) \neq 0.$$

Big idea: We know we can define analytic N-th root function $h(z)$ such that $h^N = H$ on small disc $D_r(z_0)$.

$$f(z) - w_0 = (z - z_0)^N h(z)^N = [\underbrace{(z - z_0) h(z)}_{g(z)}]^N$$

Note: $g(z_0) = 0$

$$g'(z) = 1 \cdot h(z) + (z - z_0) h'(z_0)$$

So $g'(z_0) = h(z_0) \leftarrow h(z_0)^N = H(z_0) \neq 0$

can't be zero.

Picture: