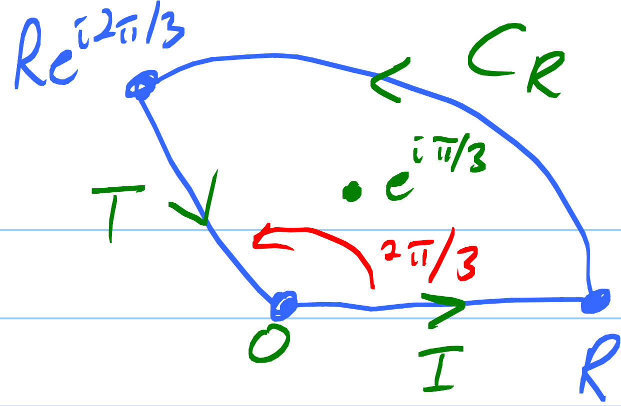
$$\int_0^\infty \frac{1}{x^3+1}\,dx$$



$I: \; z(t) = t, \quad 0 \le t \le R$

$$\int_I \frac{1}{z^3+1}\,dz = \int_0^R \frac{1}{t^3+1}\cdot 1 \cdot dt$$

$$\left| \int_{C_R} \frac{1}{z^3+1}\,dz \right| \le \left(\max_{C_R} \frac{1}{|z^3+1|} \right)\left(\frac{2\pi}{3} R \right)$$

$$|z^3+1| \ge \big| \underbrace{|z^3|}_{R^3} - |-1| \big| = R^3 - 1 \quad \text{if } R > 1.$$

$$\text{So} \quad \left| \int_{C_R} \right| \le \frac{1}{R^3-1}\cdot\left(\frac{2\pi}{3}R\right) \longrightarrow 0 \text{ as } R \to \infty.$$

$-T: \; z(t) = t\,e^{i2\pi/3} \quad 0 \le t \le R$

$$\int_T \frac{1}{z^3+1}\,dz = -\int_{-T} \frac{1}{z^3+1}\,dz = -\int_0^R \frac{1}{(\underbrace{t e^{i2\pi/3}}_{z(t)})^3+1}\, \underbrace{e^{i2\pi/3}}_{z'(t)}\,dt$$

$$= -e^{i2\pi/3} \int_0^R \frac{1}{t^3+1}\,dt$$

$$\left(\int_I + \int_{C_R} + \int_T\right) \frac{1}{z^3+1}\,dz = 2\pi i \operatorname{Res}_{e^{i\pi/3}} \frac{1}{z^3+1}$$

$$I + 0 - e^{i2\pi/3} I = 2\pi i \left(\frac{1}{3z^2}\right)\Big|_{z=e^{i\pi/3}}$$

$$I = \frac{1}{1-e^{i2\pi/3}} \cdot 2\pi i \cdot \frac{1}{3e^{i2\pi/3}} = \frac{2\pi/3}{\sin\frac{2\pi}{3}}$$

To finish, show that if f and g are analytic in a nbhd of a, and f has a <u>simple</u> zero at a, then

$$\operatorname{Res}_a \frac{g}{f} = \frac{g(a)}{f'(a)}.$$

<u>Pf</u>: $f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots$

$f(a)=0$ $\quad a_1 = \frac{f'(a)}{1!} \neq 0$

$$= (z-a)\Big[\underbrace{a_1 + a_2(z-a) + \cdots}_{F(z),\ F(a)=a_1=f'(a)}\Big]$$

$$\frac{g(z)}{f(z)} = \frac{1}{(z-a)}\left[\frac{g(z)}{\underbrace{F(z)}_{A_0 + A_1(z-a) + \cdots}}\right]$$

$$= \frac{A_0}{z-a} + A_1 + A_2(z-a) + \cdots$$

$$\underbrace{\frac{A_0}{z-a}}_{\text{Princ. Part}} \qquad \underbrace{A_1 + A_2(z-a) + \cdots}_{\text{analytic}}$$

A_0: residue

So $\mathrm{Res}_a \dfrac{g(z)}{f(z)} = A_0 = \dfrac{g(a)}{F(a)} = \dfrac{g(a)}{f'(a)}$ ✓

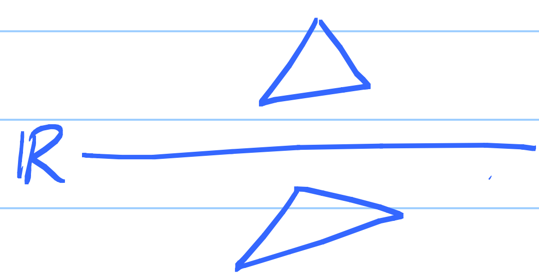
Or use Cauchy's Thm: $z^3+1 = (z - e^{i\pi/3})\,P(z)$ ← poly

$$\int_\gamma \frac{1}{z^3+1}\,dz = \int_\gamma \frac{(1/P(z))}{z - e^{i\pi/3}}\,dz$$

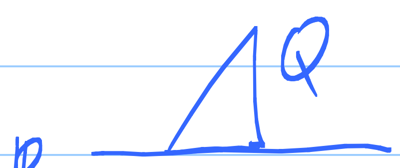
$$= 2\pi i \cdot \frac{1}{P(e^{i\pi/3})}$$

<u>Problem</u>: Suppose f is continuous on $\mathbb{C}$ and analytic on $\{z : \operatorname{Im} z > 0\}$ and $\{z : \operatorname{Im} z < 0\}$. Show that f is entire.

<u>Key</u>: Morera's!



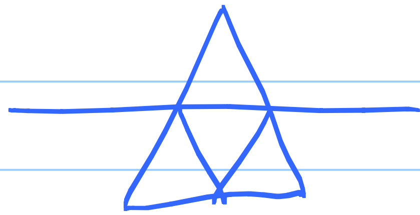
$\int = 0$ by Goursat.



Big idea: $Q : z(t)$

$Q_\varepsilon : z(t) + i\varepsilon$

$$\int_Q f\,dz = \lim_{\varepsilon \to 0} \underbrace{\int_{Q_\varepsilon} f\,dz}_{0} \quad \text{by unif cont on } D_R(0).$$

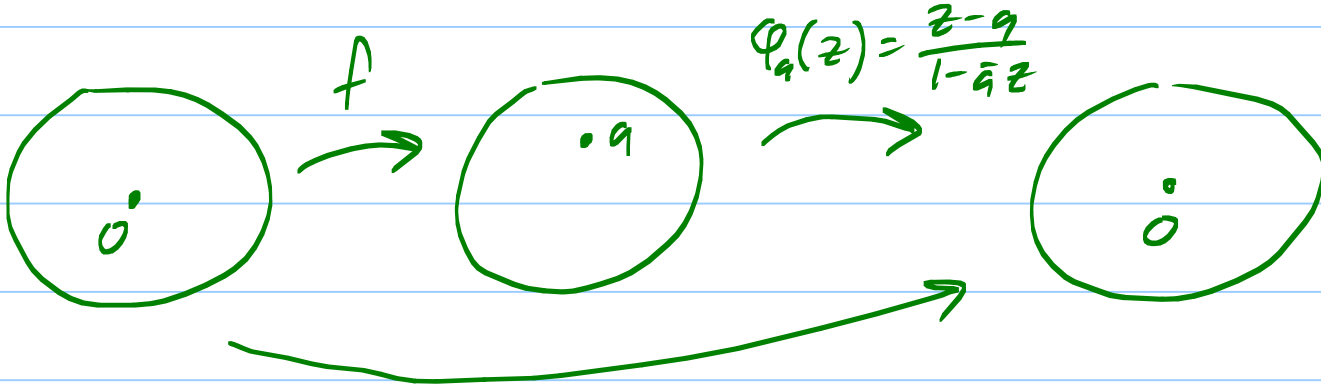


Prob $\Rightarrow$ Schwarz Reflection Principle.

Important Cor of Schwarz Lemma:

$f: D_1(0) \to D_1(0)$, analytic. Then $|f'(0)| \le 1$.

If $|f'(0)| = 1$, then $f(z) = \lambda z$, $|\lambda| = 1$.



Schwarz! F, $F(0) = 0$. $|F'(0)| \le \underline{1}$

$$\varphi_a' = \frac{1-|a|^2}{(1-\bar{a}z)^2} \qquad F'(0) = \varphi_a'(\underbrace{f(0)}_{a})\, f'(0)$$

$$= \frac{1-|a|^2}{(1-\bar{a}a)^2} \cdot f'(0)$$

Get $\left|F'(0)\right| = \left|\dfrac{f'(0)}{1-|a|^2}\right| \underset{\uparrow}{\leq} 1$

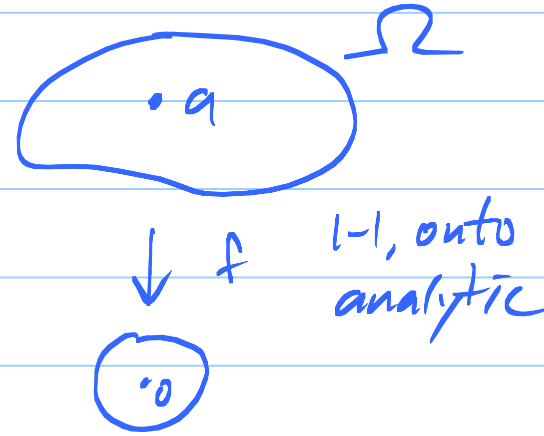
Schwarz

So $|f'(0)| \leq 1-|a|^2 \leq 1$

If $|f'(0)|=1$, see $a=0$.

$f(0)=0$. Aha! We have Schwarz! $f(z)=\lambda z$.

<u>Key to Riemann Mapping Thm</u>:



$g: \Omega \to D_1(0)$ analytic.

Then $|g'(a)| \leq |f'(a)|$.

If $|g'(a)| = |f'(a)|$, then g is 1-1 onto too!

Say we have Riemann Map f. Cor to Schwarz shows: Among all analytic maps $g: \Omega \to D_1(0)$, the R. Map maximizes $|g'(a)|$.

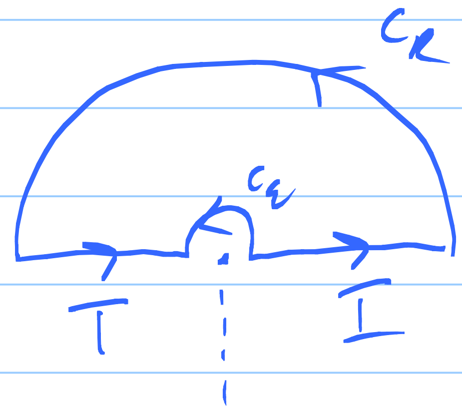
8. Schwarz plus $f(0)=0$ with mult n.

Change <u>Pf</u> of Schwarz.

$$F(z) = \begin{cases} \dfrac{f(z)}{z^n} & z \neq 0 \\ ? & z=0 \end{cases}$$

$$? = a_n = \frac{f^{(n)}(0)}{n!}$$

Max Princ Arg $\Rightarrow$ $\left|\frac{f(z)}{z^n}\right| \le 1$, etc.



$\int_{C_R} \to 0$

$\int_{C_\epsilon} \to 0$

Note: On I: $\text{Log}\, t = \ln t + i0$

On T: $\text{Log}\, t = \ln|t| + i\pi$