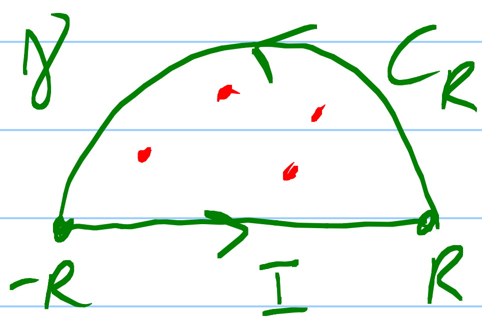
<u>Theorem</u>: Suppose P and Q are polynomials and 1) $\deg Q \geq \deg P + 2$

2) Q has no zeroes on $\mathbb{R}$.

Then
$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = 2\pi i \sum_{a \in Z_Q^+} \operatorname{Res}_a \frac{P}{Q}$$

where Z_Q^+ = the set of zeroes of Q in the UHP $= \{z : \operatorname{Im} z > 0\}$.

<u>Pf</u>:



$$\int_\gamma \frac{P}{Q}\,dz = 2\pi i \sum_{a \in Z_Q^+} \operatorname{Res}_a \frac{P}{Q}$$

for $R > R_0$.

$$\int_I \frac{P}{Q}\,dz = \int_{-R}^{R} \frac{P(t)}{Q(t)}\,dt \longleftarrow \text{want}$$

Basic Poly Est: $a|z|^{\deg P} \leq |P(z)| \leq A|z|^{\deg P}$, $|z| \geq R_P$

$b|z|^{\deg Q} \leq |Q(z)| \leq B|z|^{\deg Q}$, $|z| \geq R_Q$

$$\left|\frac{P(z)}{Q(z)}\right| \le \frac{A}{b}\,\frac{1}{|z|^{\overbrace{\underbrace{\deg Q-\deg P}_{\ge 2}}^{m}}} \quad \text{if } |z| > \operatorname{Max}(R_P, R_Q)$$

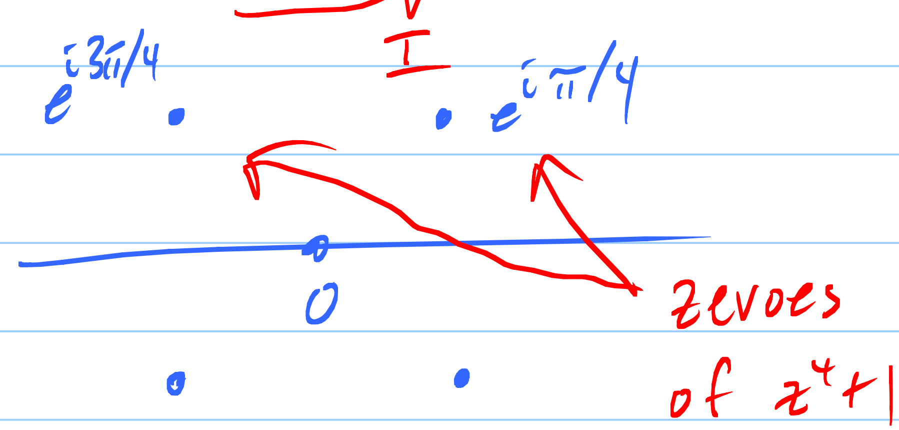
So $$\left|\int_{C_R} \frac{P}{Q}\,dz\right| \le \operatorname*{Max}_{C_R}\left|\frac{P}{Q}\right| \operatorname{Length}(C_R)$$

$$\le \frac{A}{b}\,\frac{1}{R^m}\cdot \pi R \longrightarrow 0$$

as $R\to\infty$ since $m \ge 2$.

$$\begin{array}{ccccc} \int_\gamma & & & = & 2\pi i \operatorname{Res} \\ \int_I & + & \int_{C_R} & = & \\ \downarrow & & \downarrow & & \\ \int_{-\infty}^{\infty} & + & 0 & = & \end{array}$$

EX: $$\underbrace{\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx}_{I} = 2\pi i\left(\operatorname{Res}_{e^{i\pi/4}} \frac{1}{z^4+1} + \operatorname{Res}_{e^{i3\pi/4}} \frac{1}{z^4+1}\right)$$



$$z^4+1 = \prod_{j=1}^{4}(z-r_j), \quad r_j \text{ roots of } -1.$$

$$\frac{1}{z^4+1} = \frac{1}{z-e^{i\pi/4}} \underbrace{\left(\frac{1}{\prod_{j\neq 1}(z-r_j)}\right)}_{A_0 + A_1(z-e^{i\pi/4})+\cdots}$$

$$= \frac{\overset{\text{Res}}{A_0}}{z-e^{i\pi/4}} + \text{analytic}$$

Easy way: $\quad \text{Res}_a \dfrac{g(z)}{f(z)} = \dfrac{g(a)}{f'(a)}$

if g analytic near a and f has a simple zero at a.

Simple zero: $\quad f(a)=0$

$\quad\quad f'(a)\neq 0.$

$\dfrac{1}{z^4+1}$ $\leftarrow g(z)=1$, $\leftarrow f(z)=z^4+1$, $\quad f'(z)=4z^3$

$$f'(e^{i\pi/4}) = 4e^{i3\pi/4} \neq 0$$

$$I = 2\pi i\left(\frac{1}{4(e^{i\pi/4})^3} + \frac{1}{4(e^{i3\pi/4})^3}\right)$$

$$= \frac{\pi i}{2} \left(\frac{1}{e^{i3\pi/4}} + \frac{1}{e^{i9\pi/4}} \right)$$

$$\ldots\ldots = \frac{\pi/4}{\sin \pi/4}$$