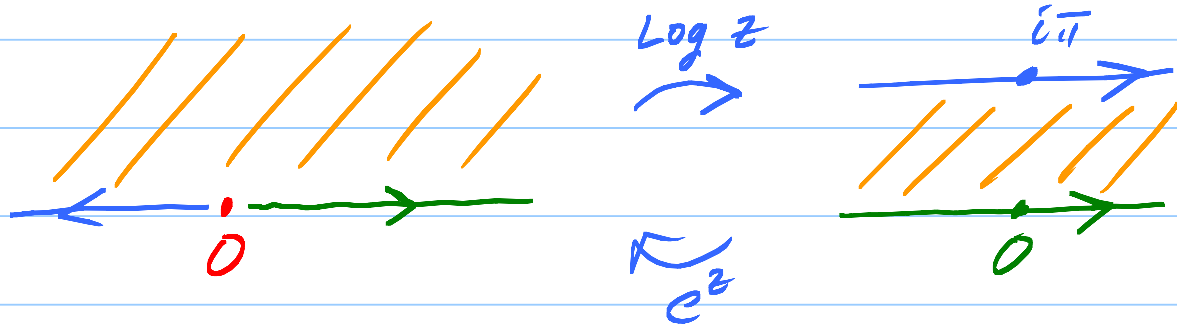
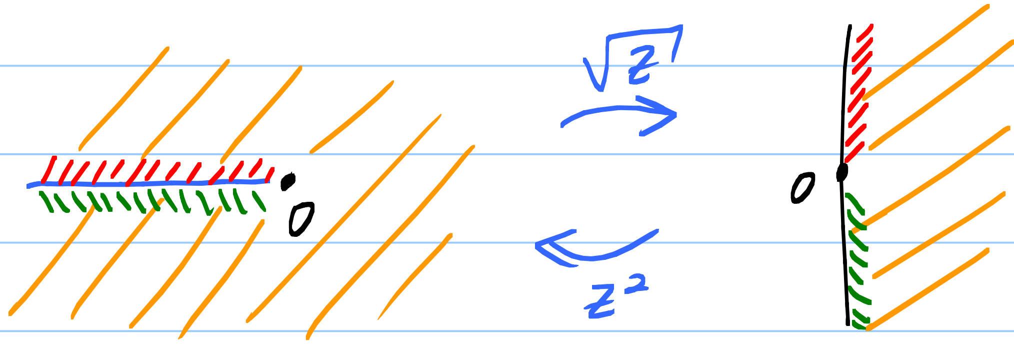
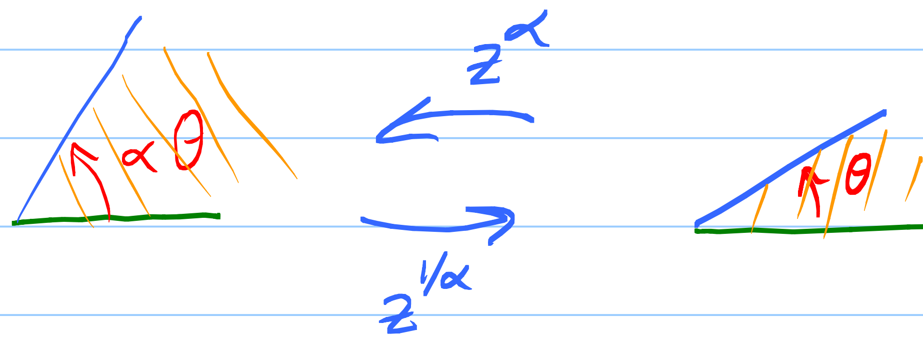
Other conformal maps



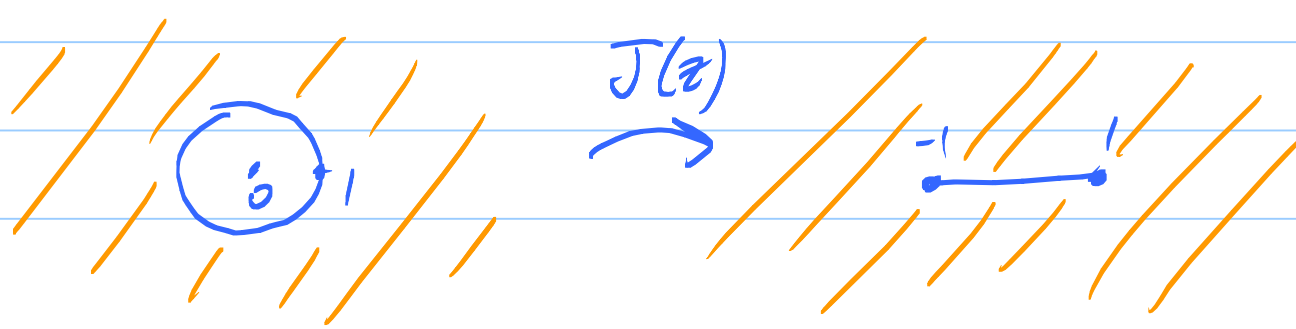


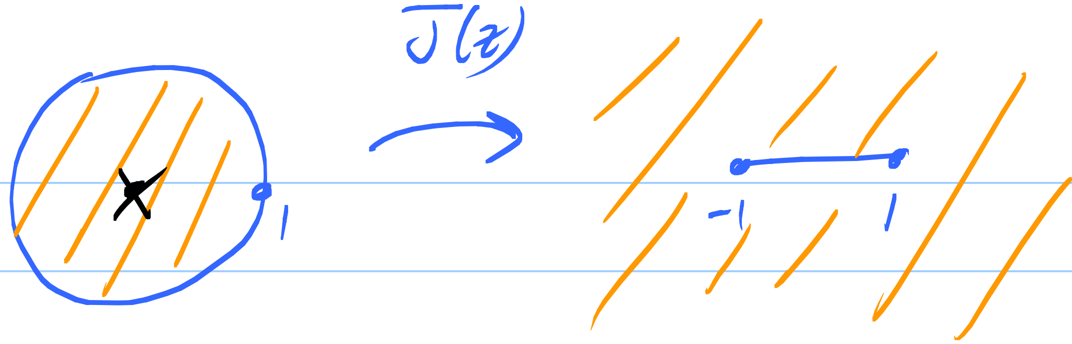
$z^{1/2} = e^{\frac{1}{2}\operatorname{Log} z}$ ← Principal branch of $z^{1/2}$

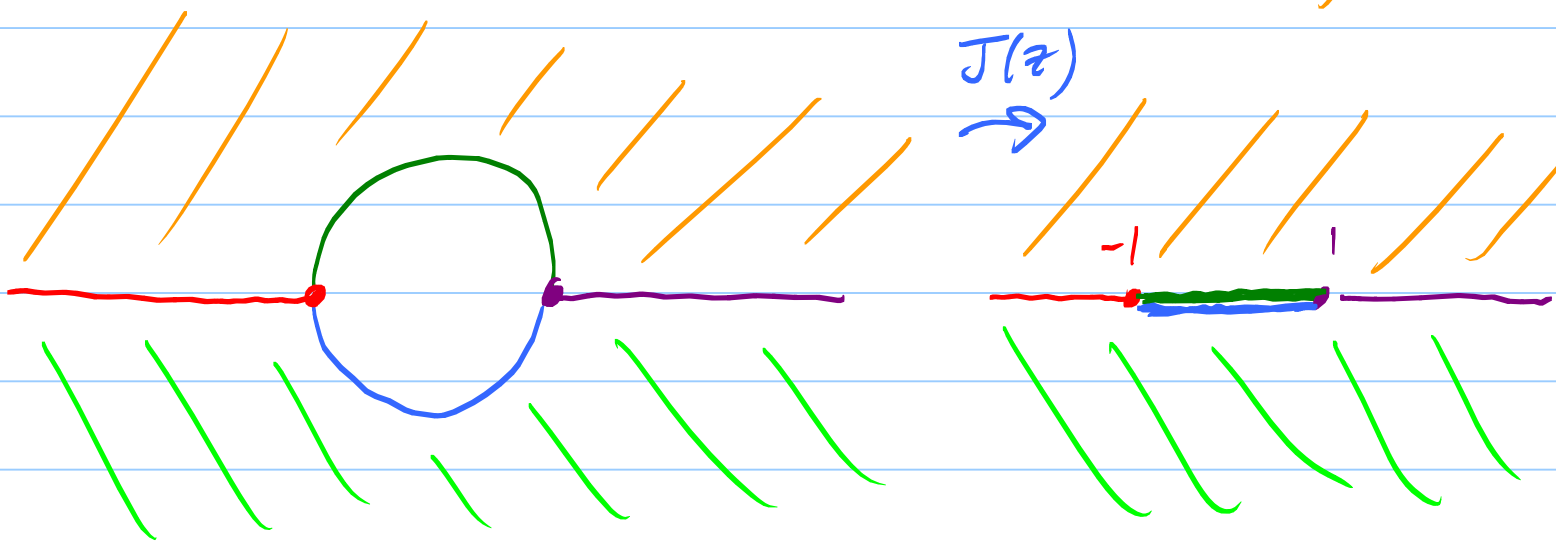


$$z^{\alpha} = e^{\alpha \operatorname{Log} z} \qquad (\alpha > 0).$$

Jukovsky Map: $J(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$

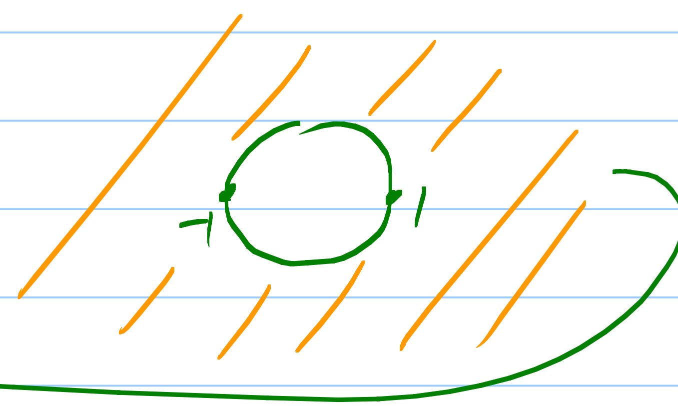


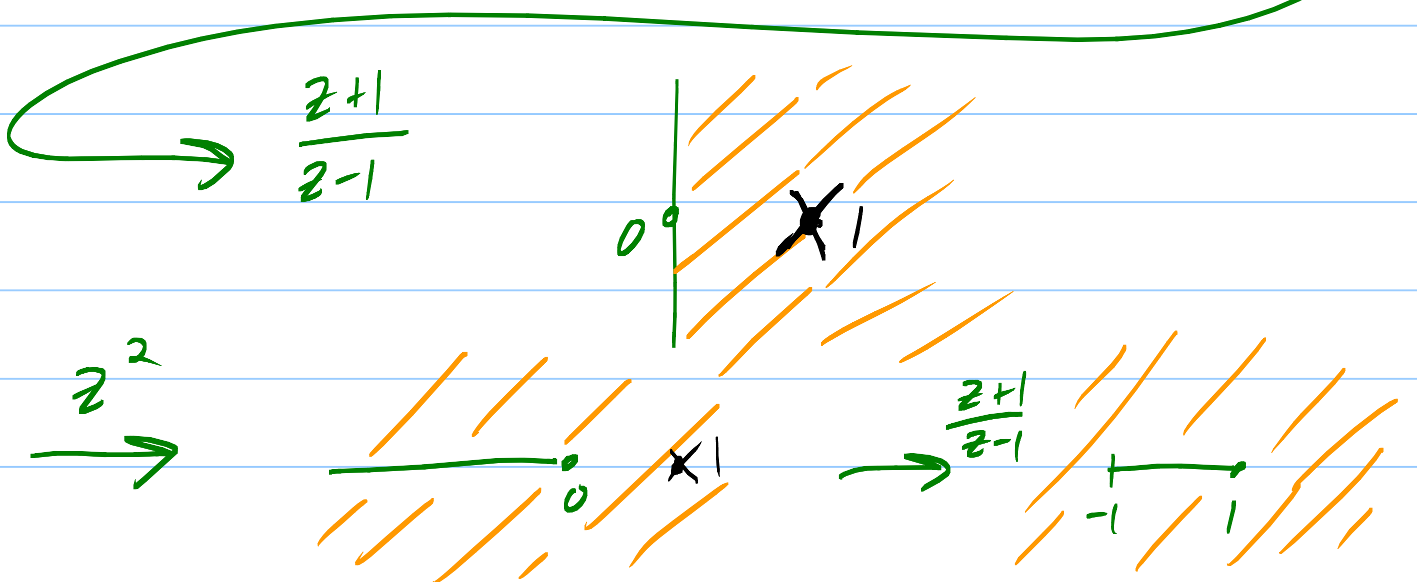




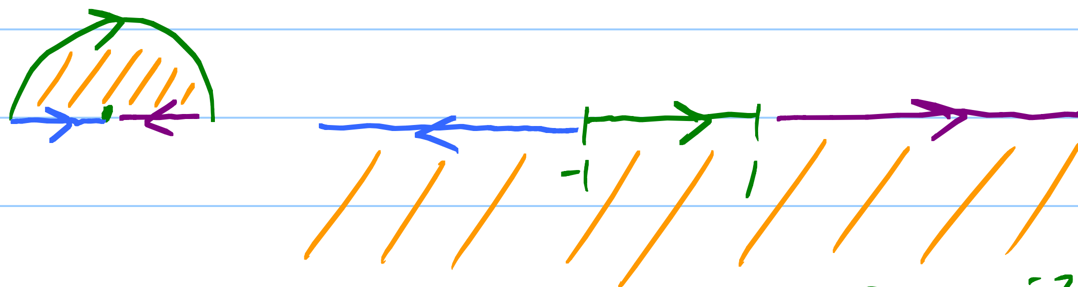
Fact: J^{-1} of horizontal lines are streamlines for 2-d flow past a cylinder.

How to cook up $J(z)$:



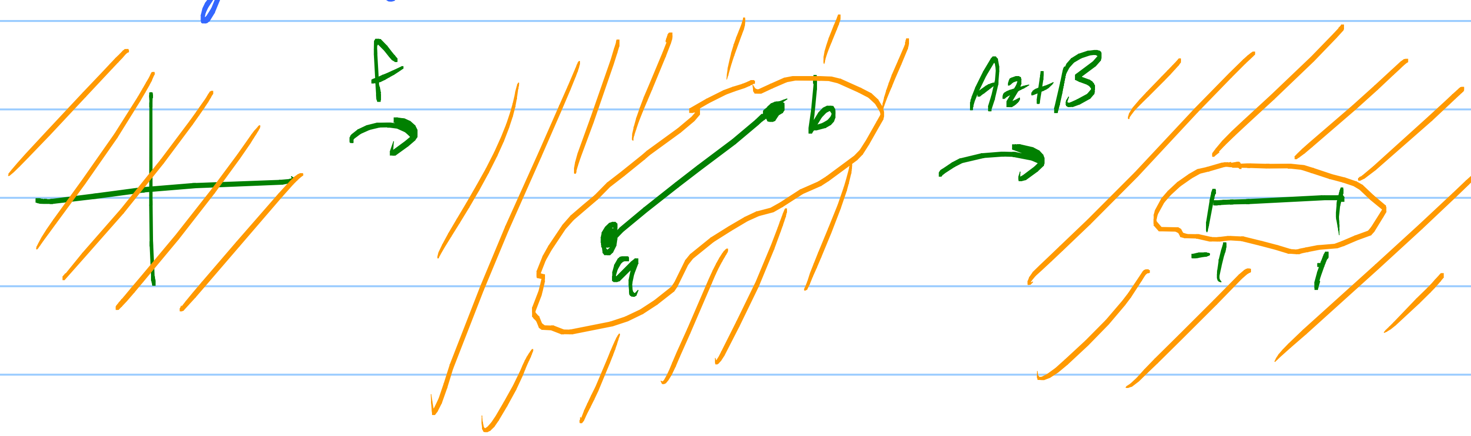


Composition is $J(z)$.



Complex Cosine: $\cos z = \dfrac{e^{iz}+e^{-iz}}{2} = J(e^{iz})$

Remark: If an entire function misses a line segment, it must be constant.

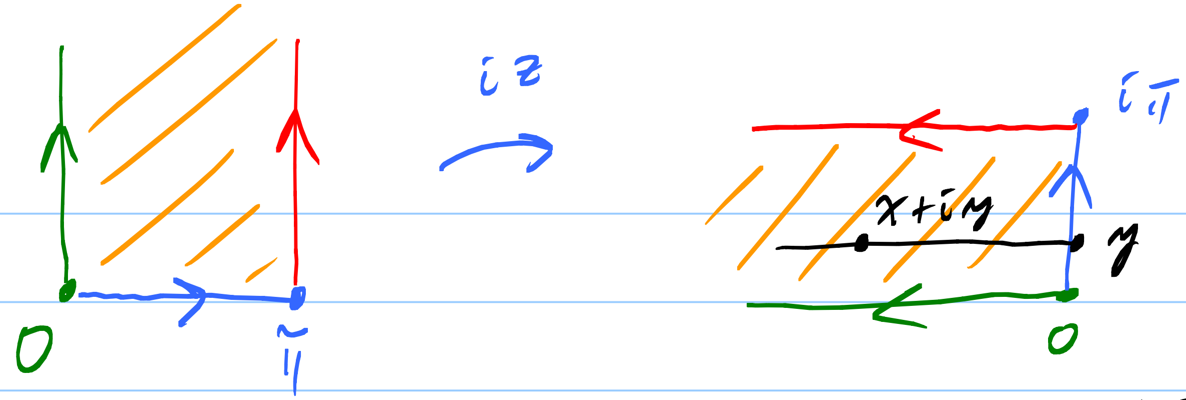


Bounded entire! Liouville's.

(Picard's Little Thm: just missing two points makes f const.)

What is J^{-1}: $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$

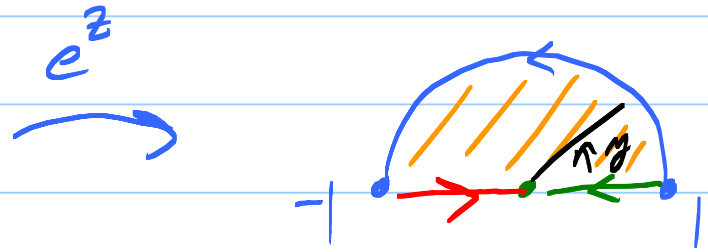
$z = w + \sqrt{w^2-1}$ careful about branch!

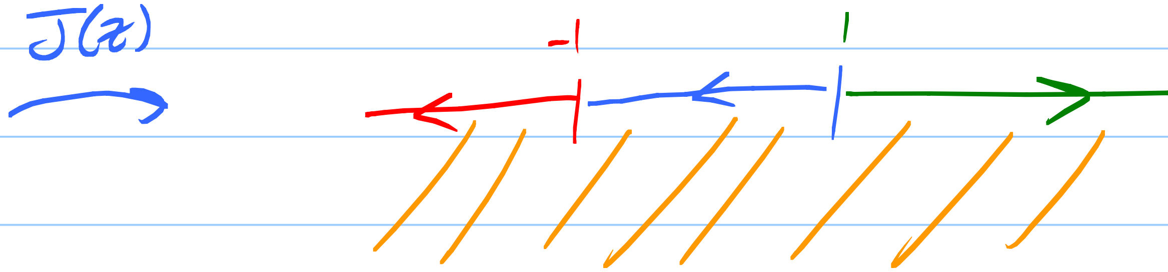


$e^{x+iy} = e^x e^{iy}$

$-\infty < x < 0$

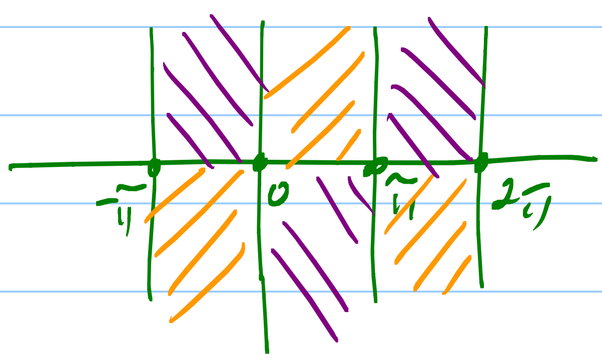
$0 < e^x < 1$

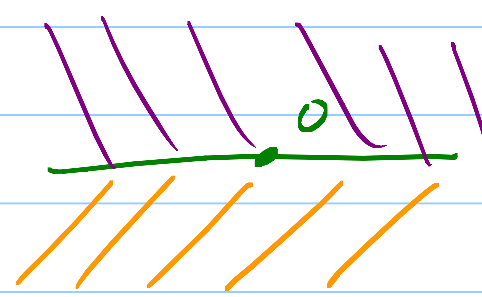




But $\cos(-z) = \cos z$

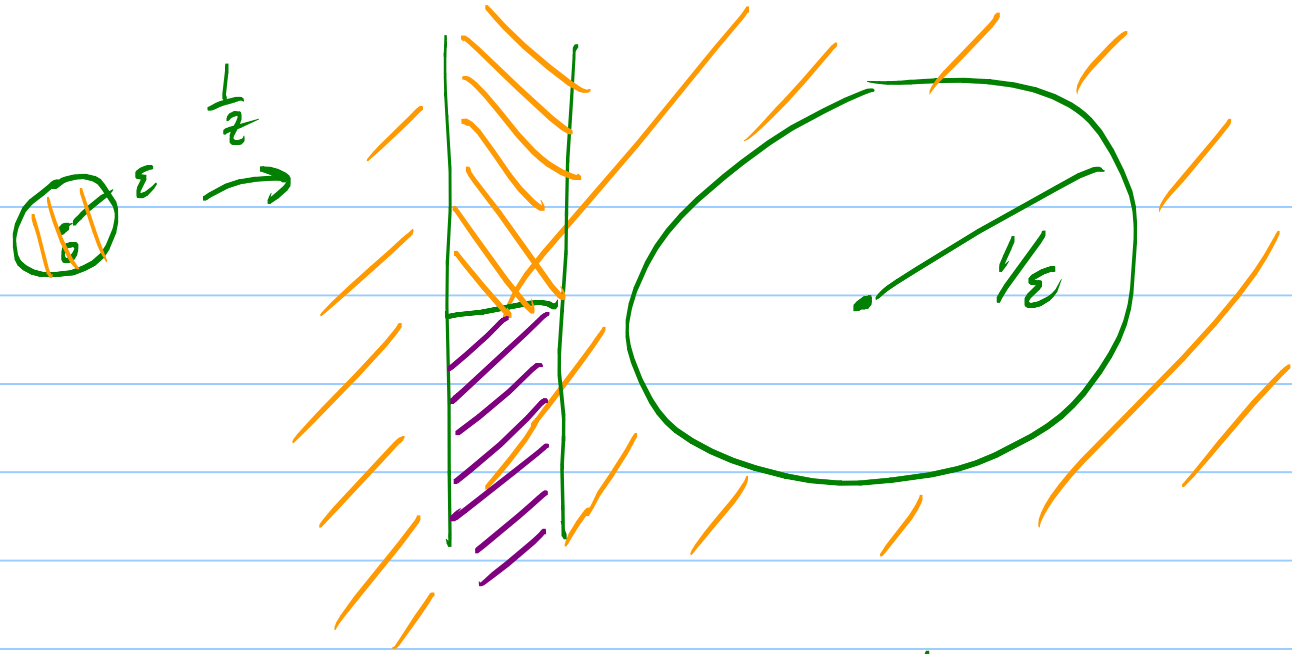
$\cos(z+\pi) = -\cos z$





Know $\sin z$ because $\sin z = \cos\left(z - \frac{\pi}{2}\right)$.

<u>Remark</u>: $\cos \frac{1}{z}$ has an essential singularity at $z = 0$.



See whole $\mathbb{C}$ gets hit by $\cos\frac{1}{z}$ on $D_\varepsilon(0)-\{0\}$ (∞-many times).

<u>Remark</u>: Picture of $\cos z$ shows how to cook up $\cos^{-1} w$.

$$w=\frac{1}{2}\left(e^{iz}+e^{-iz}\right) \leftarrow \text{mult by } e^{iz}$$

$$2we^{iz}=(e^{iz})^2+1$$

$$(e^{iz})^2-2we^{iz}+1=0$$

$$e^{iz}=\frac{2w+\sqrt{4w^2-4}}{2}=w+\sqrt{w^2-1}$$

$$z=\cos^{-1}w=\frac{1}{i}\log\left(w+\sqrt{w^2-1}\right)$$

All classical fcns are built up from e^z, its inverse, and algebraic functions! $\leftarrow$ Elementary fcns.

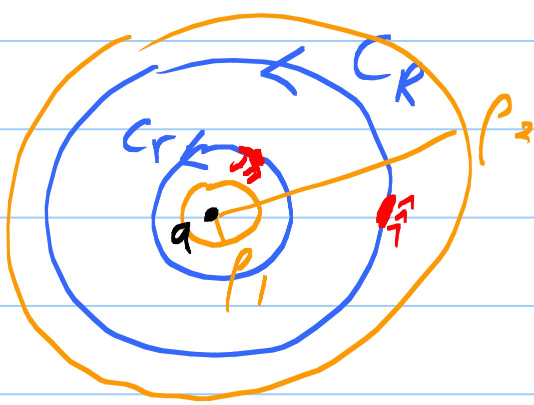
Liouville's: e^{z^2}, $\sin z^2$ do <u>not</u> have elementary antiderivatives.

<u>Lemma</u>: Cauchy Theorem for an annulus.

Suppose f is analytic on

$$A(\rho_1, \rho_2 : a) = \{z : \rho_1 < |z-a| < \rho_2\}$$

where $0 < \rho_1 < r < R < \rho_2$. Then



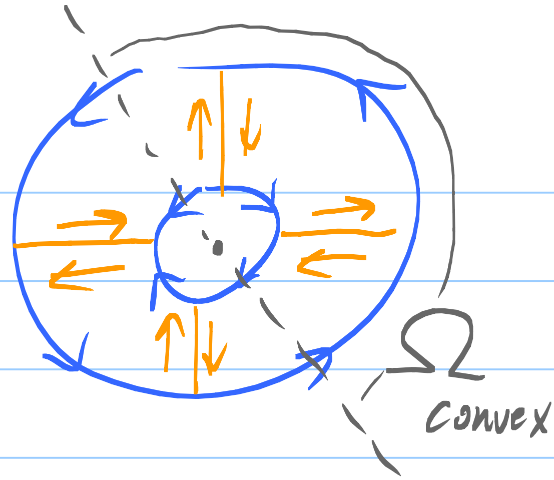
$$\left(\int_{C_R} - \int_{C_r}\right) f\,dz = 0$$

$+\int_{-C_r}$ positively oriented boundary

a bug has right legs pointing <u>out</u>.

<u>Cor</u>: $\displaystyle\int_{C_r} f\,dz = \int_{C_\rho} f\,dz = \int_{C_R} f\,dz$, $r < \rho < R$

<u>Pf of Lemma</u>: $\rho_1 = 0$ case.



$$\left(\int_{C_R} - \int_{C_r}\right) f\,dz$$

$$= \sum_1^4 \underbrace{(\text{sectors})}_{\substack{=0 \text{ by} \\ \text{Cauchy on convex.}}}$$

$\rho_1 > 0$ case; Take more cuts.