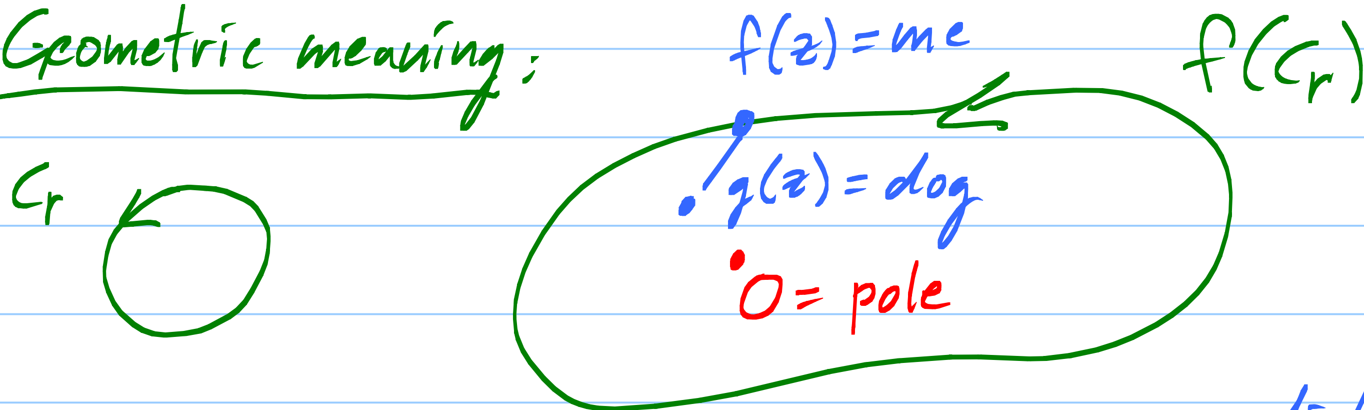
Rouché's Theorem (for a disc or a **toy region**) Suppose f and g are analytic on $D_R(a)$ and $0<r<R$. If $|f(z)-g(z)| < |f(z)|$ on $C_r(a)$, then f and g have the same number of zeroes inside $C_r(a)$ (counted with multiplicity).

Geometric meaning:



$$|f(z)-g(z)| = \text{leash length} < |f(z)| = \text{distance from me to pole}$$

#times $f(z)$ spins around 0 = #zeroes of f

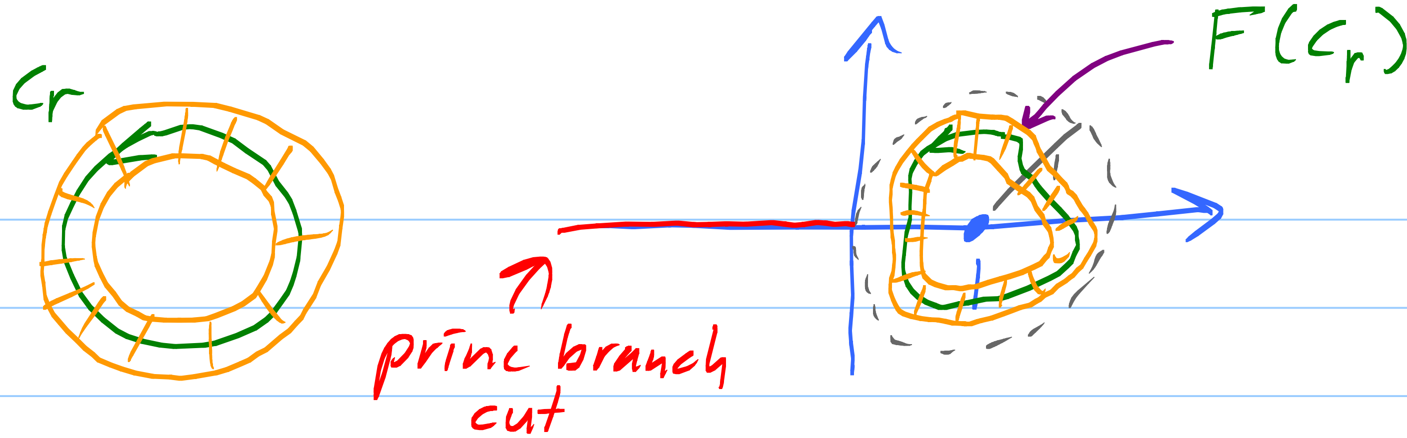
#times dog goes around pole = #times I do.

Analytic proof: $(*)\ |f(z)-g(z)| < \underbrace{|f(z)|}_{\text{no zeroes on } C_r}$ on C_r

If $g(z)=0$ on C_r, get

$|f(z)| < |f(z)|$. ↯. So g has no zeroes on C_r.

Divide $(*)$ by $|f(z)|$. Get $\left|1 - \underbrace{\frac{g(z)}{f(z)}}_{F}\right| < 1$ on C_r.



$$\int_{C_r} \frac{F'}{F}\,dz = \int_{C_r} \frac{d}{dz}(\operatorname{Log} F)\,dz = 0$$

analytic on orange collar of C_r

$$\text{But } \frac{F'}{F} = \frac{(g/f)'}{(g/f)} = \frac{g'}{g} - \frac{f'}{f}$$

$$\text{So } \underbrace{\frac{1}{2\pi i}\int_{C_r} \frac{g'}{g}\,dz}_{\#\text{ zeroes } g \text{ inside } C_r} = \underbrace{\frac{1}{2\pi i}\int_{C_r} \frac{f'}{f}\,dz}_{\#\text{ zeroes } f \text{ in } C_r} \quad \checkmark$$

__Remark__: Version of Rouché's allows finitely many poles inside C_r. Ineq $\Rightarrow$ (#zeroes) − (#poles) inside C_r are same.

__Theorem__: (Hurwicz Thm). Suppose f_n is a seq of __non-vanishing__ analytic fcns on a domain

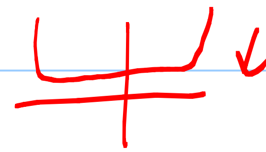
Ω that converges unif on compacts to f. (We know f is analytic.) Then, either

A) f is non-vanishing, or

B) $f \equiv 0$ on Ω.

Ex: $\Omega = D_1(0) - \{0\}$. $f_n(z) = z^n$. $f \equiv 0$.

Remark: Way false $\mathbb{R} \to \mathbb{R}$.



Pf: Suppose $f(z_0) = 0$, $z_0 \in \Omega$. If $f \not\equiv 0$, then z_0 is an isolated zero. So $\exists\ \overline{D_r(z_0)} \subset \Omega$ so that z_0 is only zero of f in $\overline{D_r(z_0)}$. On C_r:

$$|f_n(z) - f(z)| < m = \min_{C_r} |f| \leq |f(z)|$$

for $n > N$, since C_r compact, unif conv on compacts. Rouché's $\Rightarrow$ f_n and f have same number of zeroes in C_r. Oops! f has at least one, f_n has none. $\lightning$. Done!

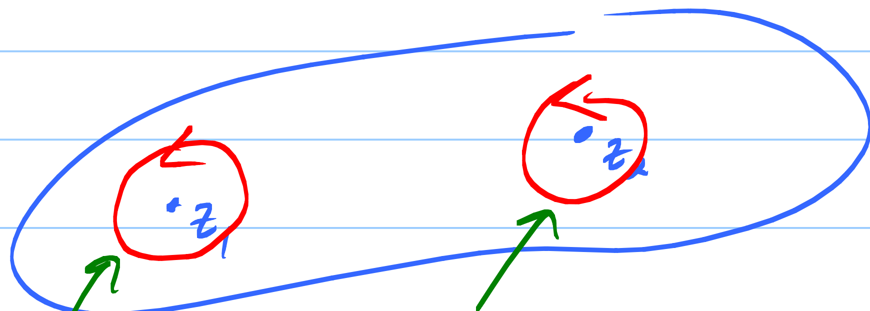
Corollary: $f_n \rightrightarrows f$. If each f_n is one-to-one analytic on Ω, then either

A) f is one-to-one, or

B) $f \equiv$ const. on Ω.

Pf: Just redo pf above. Suppose $\exists$ $z_1, z_2 \in \Omega$ with $f(z_1) = f(z_2) = w_0$

If $f \not\equiv w_0$, do this



$f(z) - w_0$ has a zero inside

So $f_n(z) - w_0$ has a zero inside too for $n > N$.

Oops! f_n not 1-1. $\lightning$.

Plan of proof of Riemann Mapping Thm:

Schwarz made us suspect that the Riemann map has maximum $|f'(a)|$ at a point a.

Let $M = \sup\{|h'(a)| : h : \Omega \to D_1(0)$, analytic, 1-1$\}$

Take seq h_n in class with $|h_n'(a)| \to M$.

Take convergent subseq. Hurwicz $\Rightarrow$

$h_n \to f$, 1-1, Schwartz $\Rightarrow$ f is onto too.

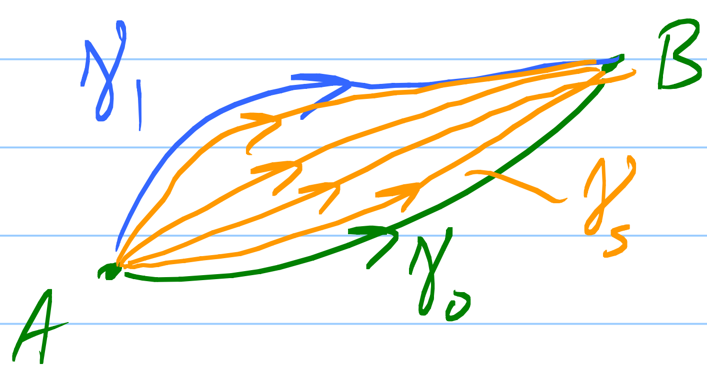
f is Riemann map.

<u>Homotopy</u>: Suppose γ_0 and γ_1 are two paths in a domain Ω from A to B,

$\gamma_j : z_j(t)$, $0 \le t \le 1$. γ_0 and γ_1 are <u>homotopic</u> in Ω (write $\gamma_0 \sim \gamma_1$ or $\gamma_0 \sim_\Omega \gamma_1$) if there is a <u>continuous</u> function $H : [0,1] \times [0,1] \to \Omega$ such that

$$\begin{cases} H(0,t) = z_0(t), & 0 \le t \le 1 \\ H(1,t) = z_1(t), & 0 \le t \le 1 \end{cases} \qquad \begin{cases} H(s,0) = A \\ H(s,1) = B \end{cases} \forall s$$

Think: $\gamma_s : z_s(t) = H(s,t)$

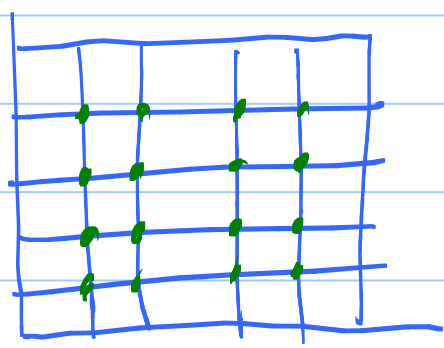


Easy fact: Given a continuous homotopy, can find a nearby piecewise $\mathcal{C}^1$ homotopy.

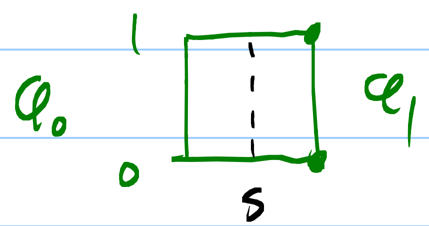
EX:

← approx continuous curve by piecewise linear curve.

Fact:



Linearize curves in the vertical direction.



$$\varphi_s = \underbrace{(1-s)\varphi_0(t) + s\,\varphi_1(t)}_{\text{convex combo}}$$

Do this on all the little squares. Make grid so fine that we stay in Ω while we do this.

Big Theorem: $\gamma_0 \sim \gamma_1$ in Ω and f analytic on Ω, then $\int_{\gamma_0} f\,dz = \int_{\gamma_1} f\,dz$.