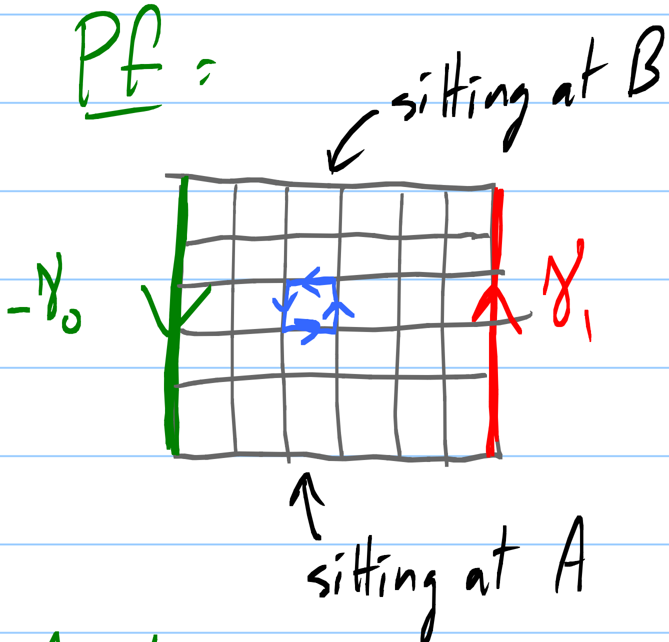
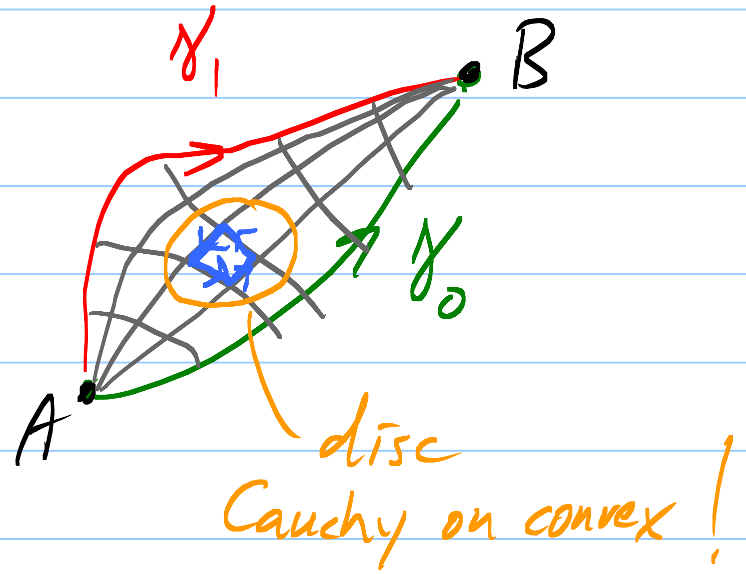
Theorem: f analytic on a domain Ω, $\gamma_0 \sim \gamma_1$. Then $\int_{\gamma_0} f\,dz = \int_{\gamma_1} f\,dz$.

Pf:





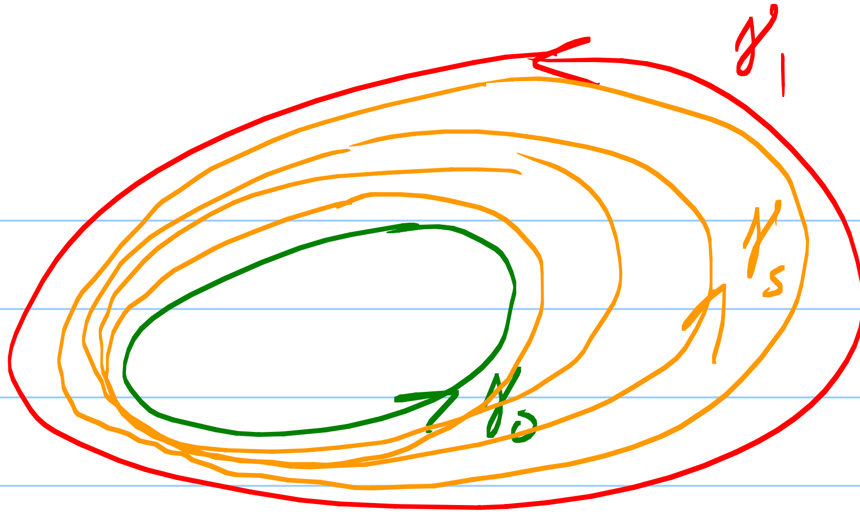
Add up little squares. Get $\left(\int_{\gamma_1} + \int_{-\gamma_0}\right) + 0 + 0 = 0$.

Def^n: Two closed curves γ_0 and γ_1 in Ω are homotopic in Ω if there is a homotopy

$H : [0,1] \times [0,1] \to \Omega$ such that

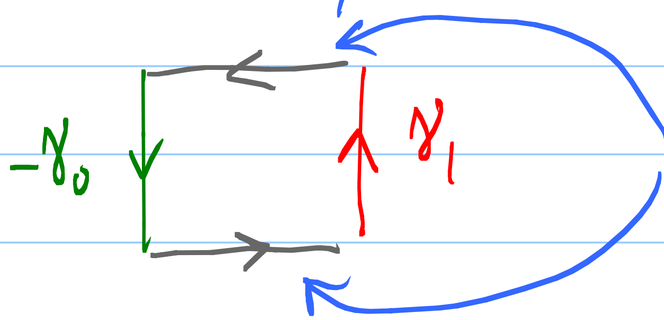
$$\begin{cases} H(0,t) = z_0(t) \leftarrow \gamma_0 \\ H(1,t) = z_1(t) \leftarrow \gamma_1 \end{cases} \quad \text{and} \quad H(s,0) = H(s,1)$$

Think $H(s,t) = z_s(t) \leftarrow \gamma_s$



Big Theorem 2: $\int_{\gamma_0} f\,dz = \int_{\gamma_1} f\,dz$ if γ_0 and γ_1 are closed and homotopic in Ω and f is analytic on Ω.

Pf:



Top cancels bottom

Def^n: A domain Ω is called simply connected if every closed curve in Ω is homotopic to a constant curve $z(t) \equiv z_0 \in \Omega$, $0 \le t \le 1$. (Say γ is "contractible to a point" in Ω.)

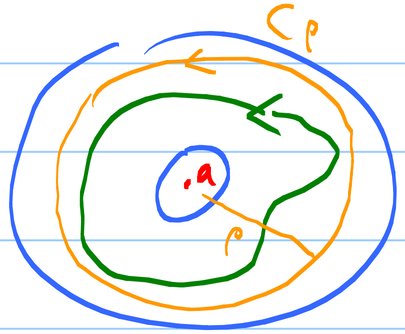
The Cauchy Theorem: Suppose f is analytic on a simply connected domain Ω. Then

$\int_\gamma f\, dz = 0$ for any closed curve γ in Ω.

Pf: $\gamma \sim z_0$. $\quad \int_\gamma f\, dz = \int_{z_0} f\, dz = 0$

$z'(t) \equiv 0$

EX: Annulus is not s.c.



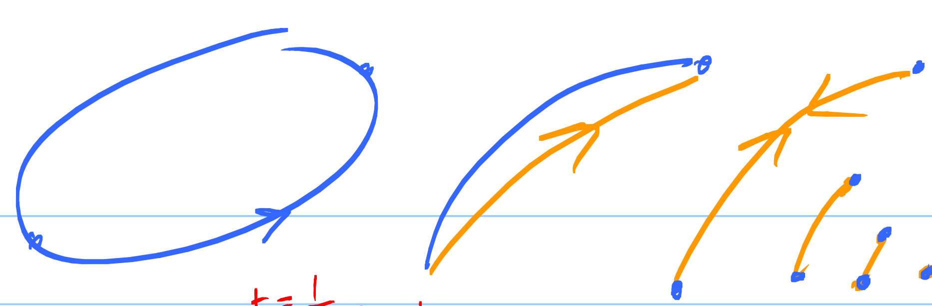
$$\int_{C_\rho} \frac{1}{z-a}\, dz = 2\pi i$$

not zero

Fact: Ω is simply connected if and only if any two curves γ_0 and γ_1 that start at same point and end at same point are homotopic.

$(\Leftarrow)$

Deform γ_0 to γ_1 and then "suck it in like spaghetti!"

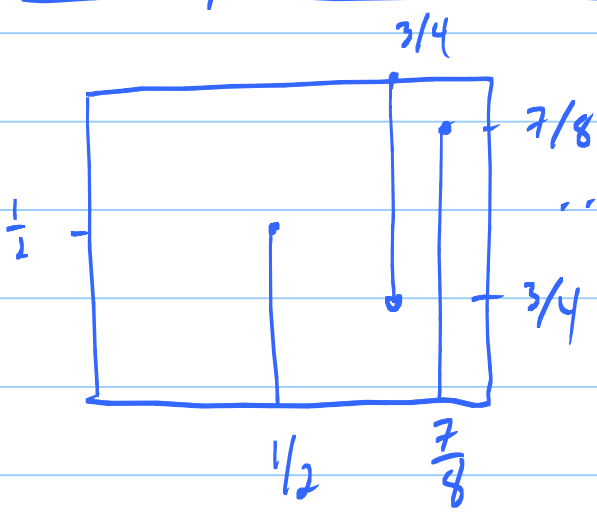


$(\Longrightarrow)$

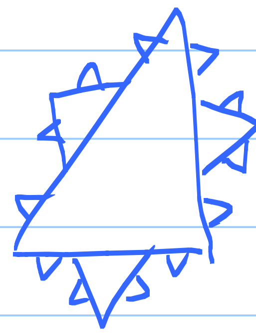
"drag" $t=0$ spot

$$\gamma = \gamma_0 \cup (-\gamma_1)$$

Examples of s.c. domains:

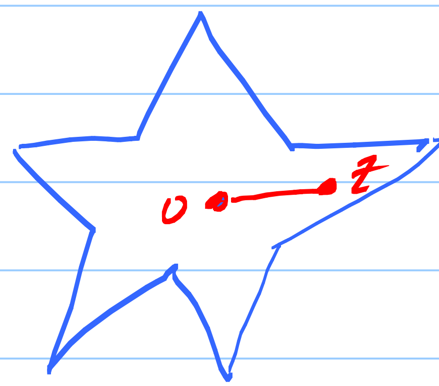


... etc.

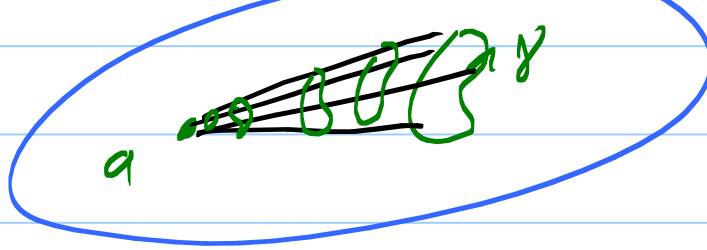


etc

Star shaped regions are s.c



Convex open sets are s.c.



Thms: Suppose f analytic on a simply connected domain Ω.

A) $\exists$ analytic F on Ω with $F' = f$;

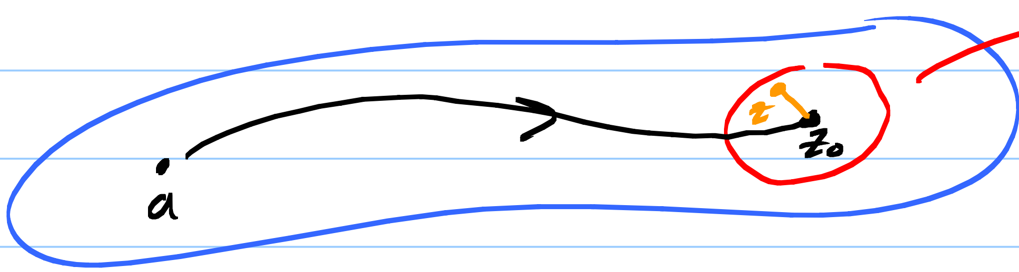
B) Furthermore, if f is non-vanishing, $\exists$ analytic G on Ω with $e^G = f$. N-th roots, too ($e^{G/N}$).

Pf: Similar to convex proof.

Define $$F(z) = \int_{\gamma_a^z} f(w)\,dw$$

curve in Ω starting at fixed a, ending at z.

$\gamma_a^z \cup (-\tilde{\gamma}_a^z)$ is closed. Cauchy $\Rightarrow$ F well defined.



restrict to disc

$$F(z) = \underbrace{\int_{\gamma_a^{z_0}}}_{\text{const.}} + \underbrace{\int_{L_{z_0}^{z}}}_{\text{know derivative } = f(z).}$$

B) $$g(z) = \int_{\gamma_a^z} \frac{f'(w)}{f(w)}\, dw$$

Then $g' = \frac{f'}{f}$. Show that correct

$G = g + \text{const.}$

<u>Theorem</u>: If u is harmonic on a s.c. domain Ω, then u has a global harmonic conj v on Ω.

Review Arzela-Ascoli Theorem. Rudin p. 158.
Ahlfors p. 219-227. Stein p. 225.