$f = u + iv$ analytic

$f' = u_x + iv_x = u_x - iu_y$

Theorem: If u is harmonic on a simply connected domain Ω, then u has a global harmonic conjugate v on Ω.

Pf: We know that $g = u_x - iu_y$ is analytic on Ω. (C-R eqns). $\exists$ analytic G on Ω with $G' = g$.

Claim: $V = \operatorname{Im} G$ is a harmonic conj for u.

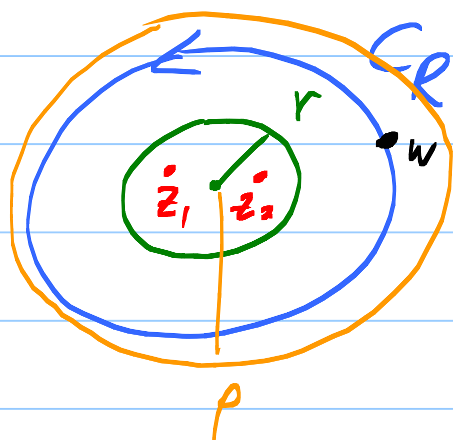
Write $G = U + iV$. $\quad G' = U_x - iU_y = \underbrace{u_x - iu_y}_{g}$

So $\nabla u = \nabla U$ on Ω. Hence $u = U + c$ on Ω. Now $u + iV = \underbrace{(U+iV)+c}_{\text{analytic}}$

So V is a harmonic conj for u on Ω.

Big Fact: Uniformly bounded analytic functions are uniformly equicontinuous on compact subsets.

Why:



f analytic on $D_\rho(a)$.

$z_1, z_2 \in D_r(a)$.

$$f(z_1) - f(z_2) = \frac{1}{2\pi i}\int_{C_R} f(w)\underbrace{\left[\frac{1}{w-z_1} - \frac{1}{w-z_2}\right]}_{(z_1-z_2)\cdot\frac{1}{(w-z_1)(w-z_2)}} dw$$

So $$|f(z_1)-f(z_2)| = |z_1-z_2| \left| \frac{1}{2\pi} \int_{C_R} \frac{f(w)}{(w-z_1)(w-z_2)}\, dw \right|$$

$$\leq \underbrace{|z_1-z_2| \underbrace{\max_{C_R} |f|}_{\leq M} \cdot \frac{1}{(R-r)^2} \cdot 2\pi R \cdot \frac{1}{2\pi}}_{\mathfrak{M}}$$

Given $\varepsilon > 0$, $\delta = \frac{\varepsilon}{\mathfrak{M}}$ works for all f with $|f| \leq M$ on $\overline{D_\rho(a)}$. $\leftarrow$ $\tilde{\mathcal{F}} = \{f : |f| \leq M$ on $D_\rho(a)$, f analytic$\}$

δ does not depend on f in $\tilde{\mathcal{F}}$.

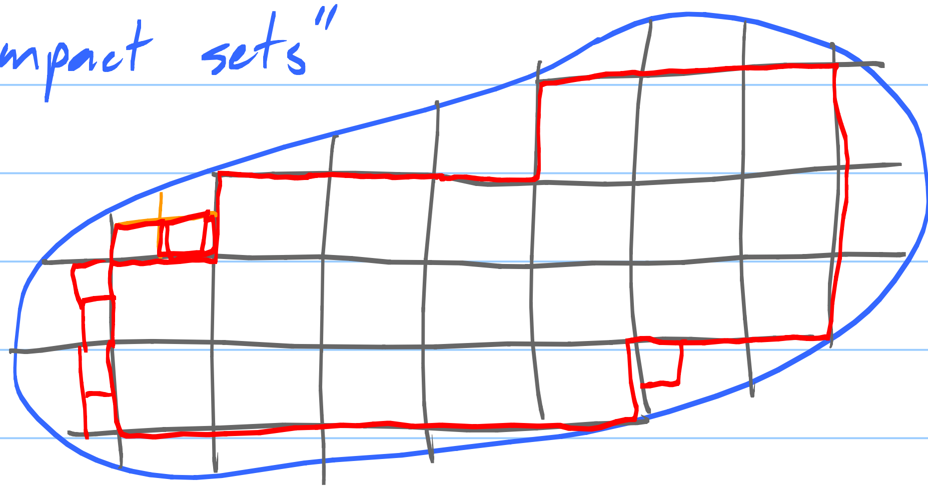
Given compact K, cover with finitely many $D_r(a)$

<u>Def^n</u>: f is uniformly bounded on a set S if $\exists$ $M > 0$ such that $|f| \leq M$ on S.

<u>Montel's Theorem</u>: Suppose Ω is a domain and f_n are analytic functions on Ω that are <u>uniformly</u> bounded on compact subsets of Ω. Then there is a subsequence that converges uniformly on compacts (to an analytic fcn f on Ω).

<u>Remark</u>: Way false $\mathbb{R} \to \mathbb{R}$. $f_n = \sin(nx)$

<u>Pf</u>: <u>Step 1</u>: Ω has an "exhaustion by compact sets"



$$K_1 = \underset{\text{side} = 1}{\text{union of closed squares}} \subset \Omega \cap D_1(0)$$

$$K_2 = \underset{\text{side} = \frac{1}{2}}{\text{union of closed squares}} \subset \Omega \cap D_2(0)$$

$\vdots$

$$K_n = \ldots \text{ side} = \tfrac{1}{n} \ldots \subset \Omega \cap D_n(0)$$

Get compact $K_1 \subset K_2 \subset K_3 \subset \cdots \subset \Omega$ and $\Omega = \bigcup_{n=1}^{\infty} K_n$.

<u>Step 2</u>: Fix a compact set $K = K_n$. Let $\{z_j\}_{j=1}^{\infty}$ be a countable dense subset. (i.e., z with rational real and imag. parts in K_n).

<u>Step 3</u>: Know f_n are unif bounded on K.

So $f_1(z_1), f_2(z_1), f_3(z_1) \ldots$ bounded

So $\exists$ conv sub seq

$$f_{1,1}(z_1),\ f_{1,2}(z_1),\ f_{1,3}(z_1),\ \ldots$$

z_2 z_2 z_2 bounded too.

Take another conv subseq :

$$f_{2,1}(z_2),\ f_{2,2}(z_2),\ f_{2,3}(z_2),\ \ldots$$

Continue. Aha! $f_{n,n}$ is a seq that converges on all z_j's ! Let $g_n = f_{n,n}$.

Step 4 : Claim : g_n is uniformly Cauchy on K, and therefore converge uniformly on K.

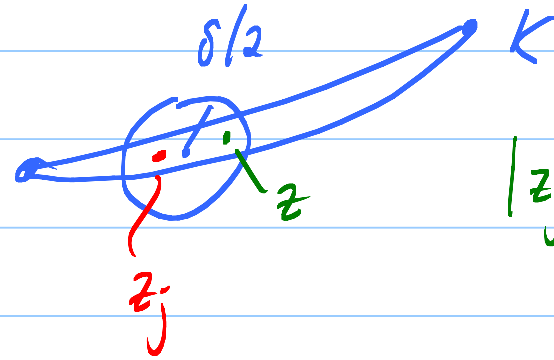
$$|g_n(z) - g_m(z)| =$$

$$|g_n(z) - g_n(z_j) + g_n(z_j) - g_m(z_j) + g_m(z_j) - g_m(z)|$$

$$\leq \underbrace{|g_n(z) - g_n(z_j)|}_{< \frac{\varepsilon}{3}} + \underbrace{|g_n(z_j) - g_m(z_j)|}_{< \frac{\varepsilon}{3}} + \underbrace{|g_m(z_j) - g_m(z)|}_{< \frac{\varepsilon}{3}}$$

First: Let $\varepsilon > 0$. Equicontinuity $\Rightarrow \exists\, \delta > 0$ such that $|g_n(z) - g_n(w)| < \frac{\varepsilon}{3}$ for $\forall n$ if $|z-w| < \delta$, $z, w \in K$.

Second: Cover K with $D_{\delta/2}(w)$, $w \in K$ and take a finite subcover. Pick one z_j in each.



$|z_j - z| < \delta$

Can make first, third terms $< \frac{\varepsilon}{3}$.

Third: $|g_n(z_j) - g_m(z_j)|$ can be made less than $\frac{\varepsilon}{3}$ if $n, m > N$ because each of these finitely many $g_n(z_j)$ converge in n.

Done!

Finally: Get $g_{1,1}, g_{1,2}, g_{1,3}, \ldots$ conv K_1.

Take sub $g_{2,1}, g_{2,2}, g_{2,3}, \ldots$ conv K_2

Take sub $g_{3,1}, g_{3,2}, \ldots$ conv K_3

etc.

$g_{n,n}$ is a subseq conv unif on all K_n.

Note: Given compact $K \subset \Omega$, $\exists\, n$ with $K \subset K_n$.