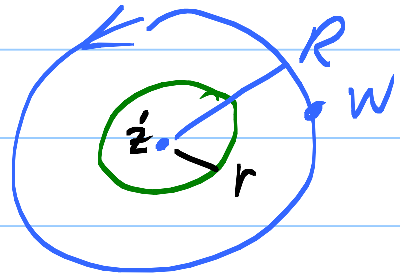
Theorem: If f_n are analytic on a domain Ω and $f_n \rightrightarrows f$ on Ω, then $f_n^{(k)} \rightrightarrows f^{(k)}$ too!

Way false $\mathbb{R} \to \mathbb{R}$. Limit f could nowhere diff.

Pf:



$$\left| f_n^{(k)}(z) - f^{(k)}(z) \right| = \left| \frac{k!}{2\pi i} \int_{C_R} \frac{f_n(w) - f(w)}{(w-z)^{k+1}}\, dw \right|$$

$$\leq \frac{k!}{2\pi} \left(\max_{C_R} |f_n - f| \right) \frac{1}{(R-r)^{k+1}}\, 2\pi R$$

C_R compact. $f_n \to f$ unif on C_R. Get $f_n^{(k)} \to f^{(k)}$ unif on $D_r(a)$. Given compact $K \subset \Omega$, cover K by $D_r(a) \subset \overline{D_R(a)} \subset \Omega$, take finite subcover. Done.

Riemann Mapping Thm: Given a simply connected domain $\Omega \neq \mathbb{C}$, there is a one-to-one analytic map $f : \Omega \to D_1(0)$ that is onto.

Proof: Let $\mathcal{F}$ be the family of analytic

$h : \Omega \to D_1(0)$ that are one-to-one.

Claim: $\tilde{\mathcal{F}}$ is not empty. (Pf. later. Easy if Ω bounded: $h(z) = z/R$, $R = \sup\{|z| : z \in \Omega\} + 1$)

Pick a point $a \in \Omega$.

Let $M = \sup\{|h'(a)| : h \in \tilde{\mathcal{F}}\}$.

Step 1: M is finite. $0 < M < \infty$.

h 1-1 $\Rightarrow$ h' non-vanish

Cauchy estimate: $|h'(a)| \le \dfrac{\overbrace{\max_{C_r}|h|}^{<1}}{r} < \dfrac{1}{r}$

So $M \le \frac{1}{r}$. (In fact, $|h'(a)| \le \dfrac{1}{\operatorname{dist}(a, b\Omega)}$)

Step 2: Take a seq $h_n \in \tilde{\mathcal{F}}$ such that $|h_n'(a)| \to M$. h_n all bounded by 1 on Ω. Montel's $\Longrightarrow$ we may assume, by taking a subseq, that $h_n \rightrightarrows f$. ← this is it!

Step 3: Today's thm shows $h_n'(a) \to f'(a)$.

So $|f'(a)| = M$. ← $M \neq 0$. So f not const.

Hurwicz #2 $\Rightarrow$ f is one-to-one.

Step 4: Claim: $f \in \widetilde{\mathcal{F}}$. Must show that

$f : \Omega \to D_1(0)$ ← into.

$$f(z) = \lim_{n\to\infty} h_n(z) \leftarrow |\,| < 1. \qquad \text{So } |f(z)| \le 1.$$

But if $|f(z)| = 1$ for $z \in D_1(0)$, then

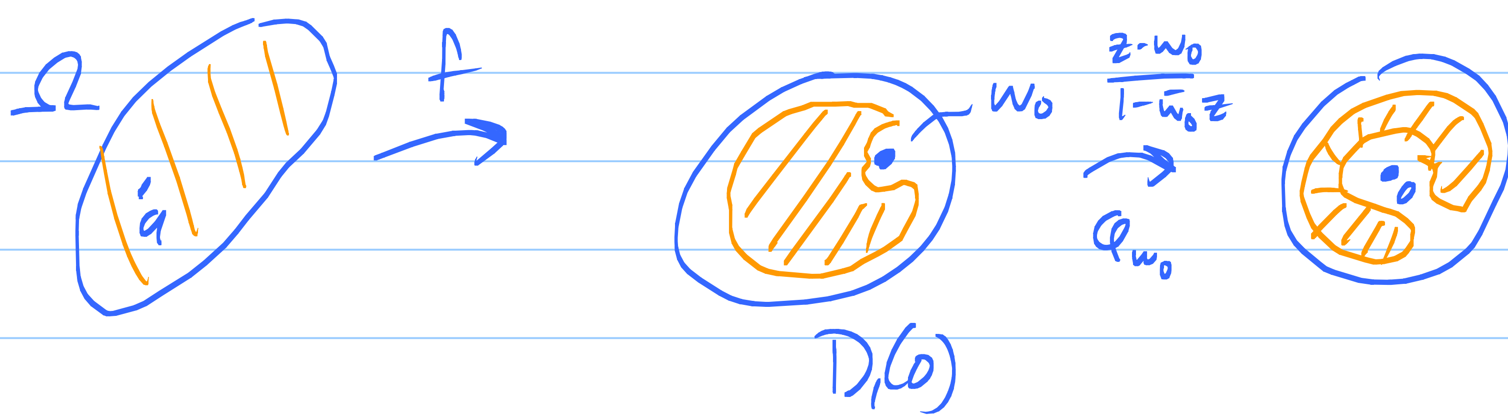
Max Princ $\Rightarrow$ f const. But it isn't.

So $|f(z)| < 1$ and $f : \Omega \to D_1(0)$.

So $f \in \widetilde{\mathcal{F}}$.

Step 5: Claim: f is onto. Suppose not.

Then $\exists\, w_0 \in D_1(0)$ that gets missed.



$\varphi_{w_0} \circ f$ is non-vanishing on Ω. Ω simp. conn. So $\exists\, g$ analytic on Ω with

$g^2 = \varphi_{w_0} \circ f$. Claim: $g \in \widetilde{\mathcal{F}}$.

g analytic ✓ maps into $D_1(0)$ ✓

one-to-one: $g(z_1) = g(z_2)$

$\Longrightarrow g(z_1)^2 = g(z_2)^2$

but $g^2 = \varphi_{w_0} \circ f$, which is 1-1. So $z_1 = z_2$.

So g is 1-1. ✓

<u>Claim</u>: $\varphi_b \circ g$ has a bigger derivative than f at a! Since $\varphi_b \circ g \in \mathcal{F}$, ↯.

Let $\tilde{f} = \varphi_b \circ g$. Check that

$$f = F \circ \tilde{f} \quad \text{where } F = \varphi_{w_0}^{-1} \circ s \circ \varphi_b^{-1}$$

where $s(z) = z^2$. Note that $F : D_1(0) \to D_1(0)$ but F is <u>not</u> one-to-one.

Corollary to Schwartz: $|F'(0)| < 1$.

[Suppose $F : D_1(0) \to D_1(0)$, analytic. Then

$|F'(0)| \le 1$. If $|F'(0)| = 1$, then $F(z) = \lambda z$, $|\lambda| = 1$ and so F is 1-1.]

Finally, $f'(a) = F'(\underbrace{\tilde{f}(a)}_{0})\,\tilde{f}'(a)$

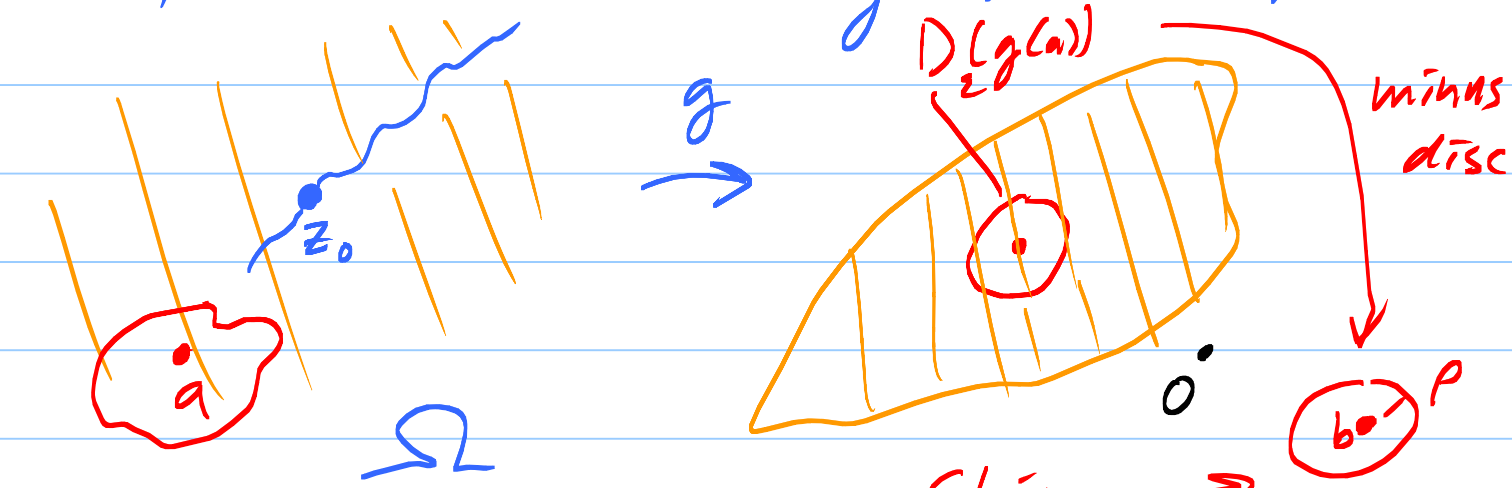
$$|f'(a)| = \underbrace{|F'(0)|}_{<1}\,|\tilde{f}'(a)| < |\tilde{f}'(a)|$$

Oops! $\tilde{f} \in \mathcal{F}$ and $|\tilde{f}'(a)| > M$. ↯. So no such w_0 exists. So f is onto. f is a Riemann Map!

<u>Last piece of proof:</u> Show $\mathcal{F} \neq \emptyset$.

$\Omega \neq \mathbb{C}$. So $\exists\, z_0 \in \mathbb{C} - \Omega$.

So $z - z_0$ is non-vanishing analytic fcn on Ω. Ω simp. conn.. So $\exists\, g$ analytic on Ω with $g^2(z) = z - z_0$.



Claim: ⟶ missed

$g(z_1) = g(z_2) \implies g(z_1)^2 = g(z_2)^2 = z_1 - z_0 = z_2 - z_0$ ⑥

$\implies z_1 = z_2.$

$\frac{\rho}{z-b}$ takes outside of $D_\rho(b)$ 1-1 into unit disc. Compose to get 1-1 analytic <u>into</u> unit disc.