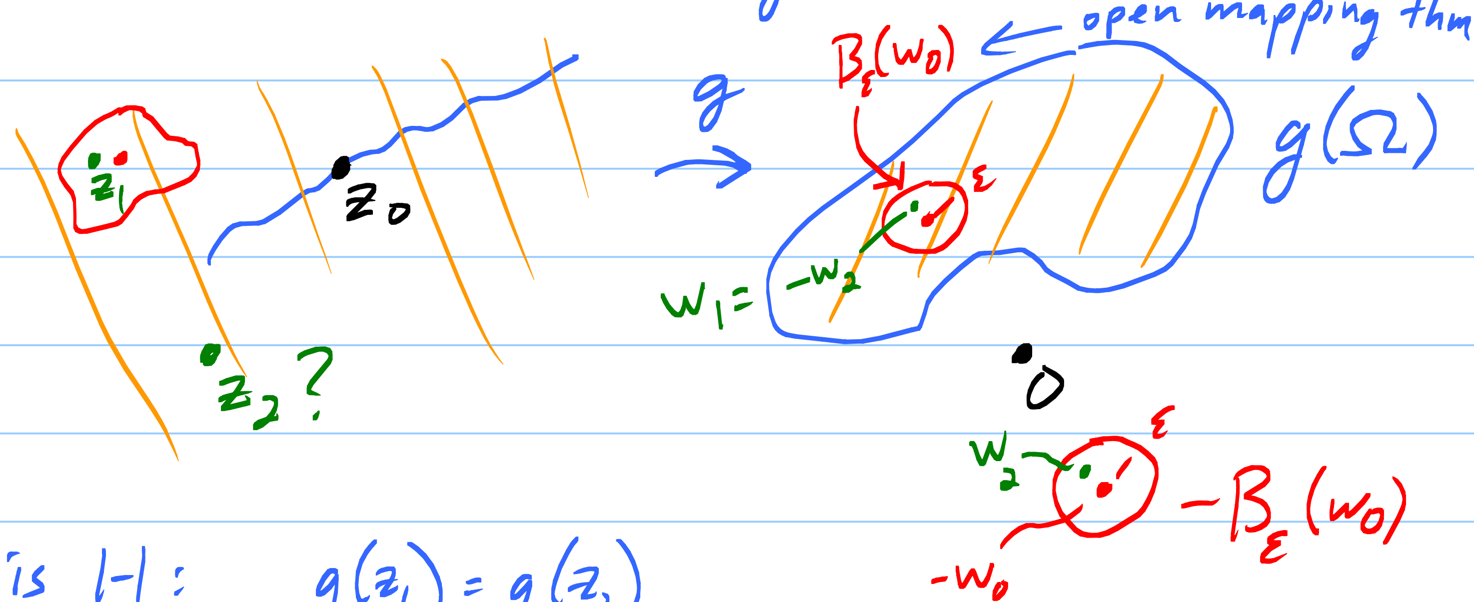
$\Omega \neq \mathbb{C}$ simply connected. $\exists$ analytic <u>one-to-one</u> map <u>into</u> $D_1(0)$.

Pick $z_0 \in \mathbb{C} - \Omega$. $\quad g(z)^2 = z - z_0$



g is 1-1:

$$g(z_1) = g(z_2)$$
$$g(z_1)^2 = g(z_2)^2$$
$$z_1 - z_0 = z_2 - z_0 \implies z_1 = z_2$$

$$\left.\begin{array}{l} g(z_1) = -w_2 \\ g(z_2) = w_2 \end{array}\right\} \quad \begin{array}{l} g(z_1)^2 = w_2^2 \\ g(z_2)^2 = w_2^2 \end{array} \quad \text{oops! } z_1 = z_2 \ \lightning.$$

Finally, compose with $\frac{\varepsilon}{z + w_0}$ to get 1-1 <u>into</u> $D_1(0)$.

<u>Remarks</u>: a) If f is a Riemann Map, then $f(a) = 0$. [$|f'(a)| = \max$]. [Otherwise, compose with a Möbius and make $|h'(a)|$ even bigger. $\lightning$]

b) If f and $\tilde{f}$ are two Riemann maps for a, then $f = \lambda \tilde{f}$ where $|\lambda| = 1$. (Schwarz).

c) The Riemann map is the one with $f'(a) \in \mathbb{R}^+$.

Koebe's Proof: Step 1. Cook up 1-1 analytic map into $D_1(0)$ as above. If onto, done! Play Schwarz Lemma – $\sqrt{\ }$ game at end of proof of RMT to cook up $f_2 : \Omega \to D_1(0)$ analytic, 1-1 with $|f_2'(a)| > |f_1'(a)|$. Repeat! Koebe showed $f_n \to$ Riemann Map.

Harmonic Functions: u C^2-smooth and

$\Delta u \equiv 0$.

1) We know that $u = \mathrm{Re}\, f$ locally, and on simply connected domains and discs.

2) Harmonic functions satisfy the Averaging Property:

$$\underbrace{u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + \rho e^{i\theta})\, d\theta}_{\text{real part of Ave Prop for } f.}$$

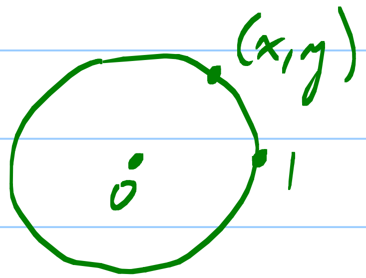
Why: $u = \mathrm{Re}\, f \leftarrow$ analytic.

Big Fact: If u is continuous and satisfies the averaging property on a domain Ω, then

u is harmonic on Ω.

Dirichlet Problem on a disc: Given a continuous fcn $\varphi(\theta)$ on $[0,2\pi]$ such that $\varphi(0)=\varphi(2\pi)$, find a harmonic function on $D_1(0)$ that is continuous on $\overline{D_1(0)}$ such that $u(e^{i\theta}) = \varphi(\theta)$.

Step 1: Solution is easy if φ is "polynomial".



$$\varphi(x,y) = \sum a_{nm} x^n y^m$$

$$x = \frac{z+\bar z}{2} \qquad y = \frac{z-\bar z}{2i}$$

$$\varphi(z,\bar z) = \sum c_{nm} z^n \bar z^m$$

Key: $\overbrace{z = \frac{1}{\bar z}}^{1}$ on unit circle, $\overbrace{\bar z = \frac{1}{z}}^{2}$ too.

$$(e^{-i\theta}) = \frac{1}{e^{i\theta}} \quad \checkmark$$

Big idea: $z^n \bar z^m$

$$\begin{cases} m > n, \text{ use } 1: \\ z^n \bar z^m = \left(\frac{1}{\bar z}\right)^n \bar z^m = \bar z^{m-n} \end{cases}$$

$$\begin{cases} m < n, \text{ use } 2: \\ z^n \bar z^m = z^n \left(\frac{1}{z}\right)^m = z^{n-m} \end{cases}$$

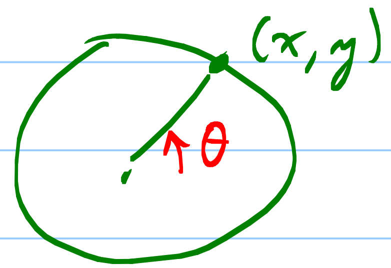
Aha! $\sum c_{nm} z^n \bar{z}^m = P(z) + \overline{Q(z)}$ ← on $C_1(0)$

P, Q: polys

The function on right solves D. Prob.!

Fejér's Thm: Suppose φ is a 2π-periodic continuous fcn on $\mathbb{R}$. Given $\varepsilon > 0$, there is a trig poly $\sum_{n=-N}^{N} a_n e^{in\theta}$ that approximates φ within ε on $[0, 2\pi]$.

Notice that:



$$e^{in\theta} = (e^{i\theta})^n = (x+iy)^n$$

$$e^{-in\theta} = (e^{-i\theta})^n = (x-iy)^n$$

Trig polys are polys in x and y on $C_1(0)$!

Solution to D. Prob: Given φ, approximate φ by trig polys $\varphi_N \to \varphi$ uniformly. Get harmonic polys u_N solve D. Prob. for φ_N.

Claim: $u_N \to u$ ← solves D. Prob.

Pf: Max Princ $\Rightarrow u_N$ conv uniformly on $\overline{D_1(0)}$ to u. $u(e^{i\theta}) = \varphi(\theta)$. ✓

Lemma: Uniform limits of harmonic fcns are harmonic.

So u is harmonic. ✓

Theorem: If u is continuous on Ω and satisfies Ave Prop., then u is harmonic.

Pf: Can restrict to a disc $\overline{D_r(a)} \subset \Omega$.

Solve D. Prob on $D_r(a)$: Get $\tilde{u}$ harmonic on $D_r(a)$, cont. on $\overline{D_r(a)}$, and $\tilde{u} = u$ on $C_r(a)$.

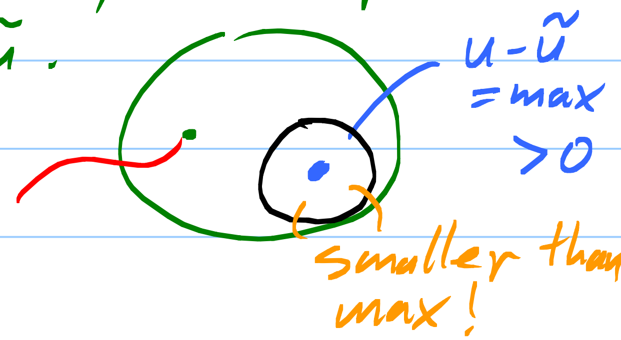
Claim: $\tilde{u} \equiv u$ on $D_r(a)$.

Pf: $\tilde{u}$ is harmonic. So Ave Prop. holds.

Ave Prop holds for u by assumption.

So Ave Prop holds for $u - \tilde{u}$.

$u - \tilde{u} > 0$



Oops! $u-\tilde{u}$ must be ≤ 0.

Repeat for $\tilde{u}-u$. So $\tilde{u}-u \le 0$ too.

Conclude $\tilde{u} \equiv u$.

Fact: Disc: Solutions to D. Prob with rational boundary data are also rational.

Thm: (B, Ebenfeld, Khavinson, Shapiro). Disc only domain with this property. [Rational data $\to$ rational sol^n.]

Beautiful fact: A domain bounded by a closed quadratic: $\Omega = \{r(x,y) < 0\}$,

$r(x,y) = ax^2 + bxy + cy^2 + dx + ey + f$ (ellipse or circle) satisfies the polynomial data $\to$ polynomial D. Prob. Sol^n.

Pf: P_N = polys in x,y of degree N.

Fisher map: $p \longmapsto \Delta(rp)$ maps $P_N \circlearrowleft$. Linear!

order 2, degree 2

Claim: F. map is 1-1: $\Delta(rp) = 0 \implies$ rp harmonic. But $r=0$ on $b\Omega$.

Max Princ $\implies$ $rp \equiv 0 \implies p \equiv 0$. ✓

$\tilde{\Delta} : P_N \xrightarrow[\text{linear}]{} P_N$ 1-1. Finite Dim'l vector space fact. It must be onto!

So, given poly q, $\exists\ p$ with

$$\Delta(rp) = \underbrace{\Delta q}_{\text{poly}}$$

$q - rp$ solves D. Prob. with boundary data q.