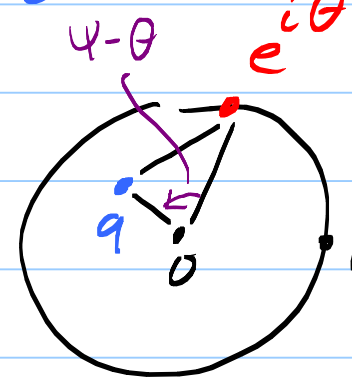
Poisson Integral:

$$u(a) = \int_0^{2\pi} P(a,\theta)\, u(e^{i\theta})\, d\theta$$



$$P(a,\theta) = \frac{1}{2\pi} \frac{1-|a|^2}{|e^{i\theta}-a|^2}$$

$$P(\underbrace{re^{i\psi}}_{a},\theta) = \frac{1}{2\pi} \frac{1-r^2}{1-2r\cos(\psi-\theta)+r^2} \leftarrow \text{law of cosines}$$

$$P(a,\theta) = \frac{1}{2\pi} \mathrm{Re}\left[\frac{e^{i\theta}+a}{e^{i\theta}-a}\right]$$

Fact: Poisson Formula holds for u continuous on $\overline{D_1(0)}$ and harmonic in $D_1(0)$.

Pf: $u(rz)$, $r<1$, is harmonic on $D_{1/r}(0)$ where $\frac{1}{r} > 1$. Poisson Formula holds for $u(rz)$. Let $r \nearrow 1$. [u cont on $\overline{D_1(0)} \Rightarrow u$ unif cont $\Rightarrow u(rz) \to u(z)$ uniformly.]

Schwarz Theorem: Poisson formula gives solⁿ to the Dirichlet Prob with boundary data $U(\theta)$.

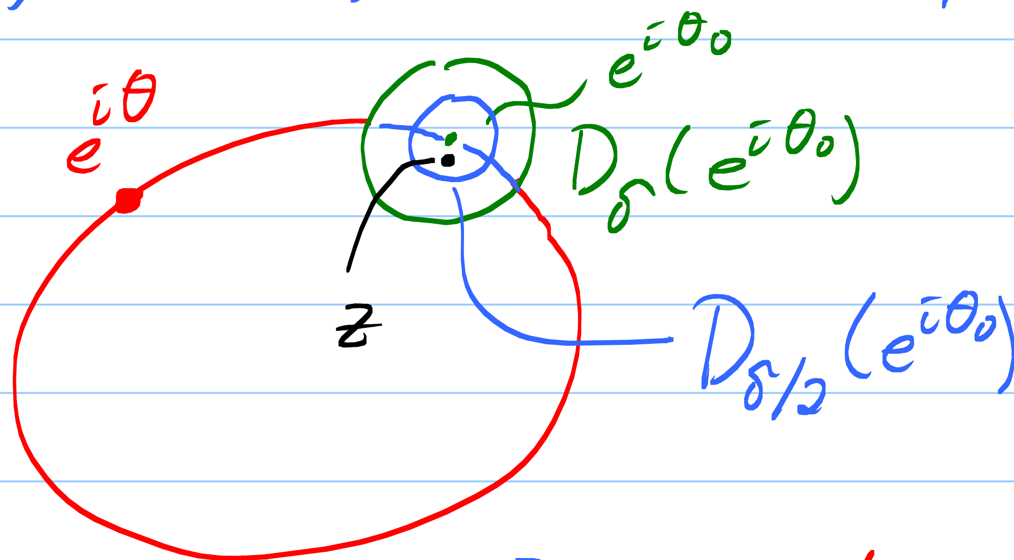
[Let $u(e^{i\theta}) = U(\theta)$ in formula.]

Three key properties of $P(a,\theta)$:

1) $P(a,\theta) > 0$ if $a \in D_1(0)$, $0 \le t \le 2\pi$.

2) $\displaystyle\int_0^{2\pi} P(a,\theta)\, d\theta = 1$ for all $a \in D_1(0)$.

3)



Fix $\delta > 0$.

$I_\delta = \{\theta : e^{i\theta} \notin D_\delta(e^{i\theta_0})\}$ ← red

If $z \in D_{\delta/2}(e^{i\theta_0}) \cap D_1(0)$ and $\theta \in I_\delta$,

then $P(z,\theta) \le \frac{4}{\pi\delta^2}(1-|z|)$.

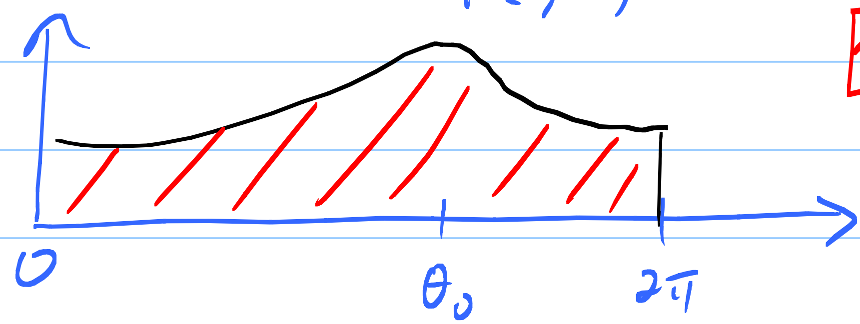
4) $P(z,\theta)$ is harmonic in z.

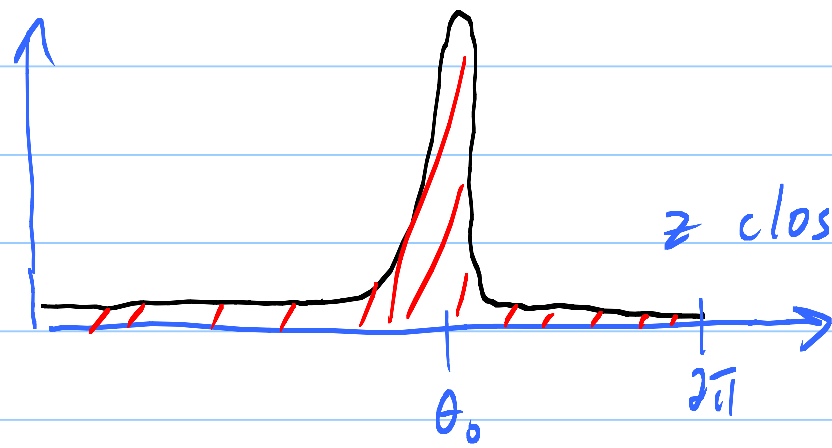
Proof: 1) Look at formula.

2) Poisson Formula for $u \equiv 1$.

4) $P(z,\theta) = \frac{1}{2\pi} \operatorname{Re}\left[\dfrac{e^{i\theta}+z}{e^{i\theta}-z}\right]$ ← analytic in z

Meaning of 3:





z close to boundary in $D_{\delta/2}(e^{i\theta_0})$

Pf of 3:

$$P(z,\theta) = \frac{1-|z|^2}{2\pi|e^{i\theta}-z|^2} \le \frac{1}{2\pi}\,\frac{(1-|z|)\underbrace{(1+|z|)}_{\le 2}}{\left(\frac{\delta}{2}\right)^2} \le \frac{4}{\pi\delta^2}(1-|z|) \checkmark$$

Pf of Schwarz: Step 1: u is harmonic on $D_1(0)$:

$$u(z) = \operatorname{Re} \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\,\tilde{U}(\theta)\,d\theta$$

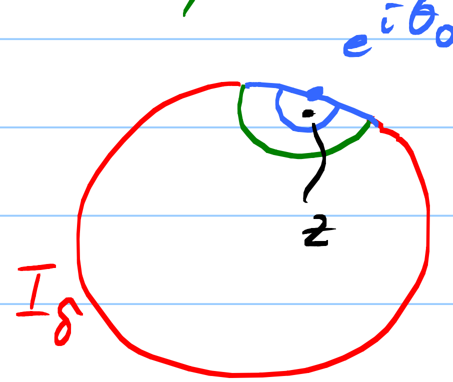
$$= \mathrm{Re}\left[\int_0^{2\pi} \frac{\frac{1}{2\pi} U(\theta) \frac{1}{i}}{e^{i\theta} - z} \cdot i e^{i\theta} d\theta \right.$$

$$\left. + z \int_0^{2\pi} \frac{\frac{1}{2\pi} U(\theta) \frac{1}{i e^{i\theta}}}{e^{i\theta} - z} i e^{i\theta} d\theta \right]$$

$$= \mathrm{Re}\left[\int_C \frac{\varphi(w)}{w-z} dw + z \int_C \frac{\psi(w)}{w-z} dw \right]$$

Prob 1 from HWK 2 shows [ ] is analytic.

<u>Step 2</u>: u extends continuously to $\overline{D_1(0)}$ and $u(e^{i\theta}) = U(\theta)$.



$$u(z) - U(\theta_0) = \int_0^{2\pi} P(z,\theta)\left[U(\theta) - U(\theta_0)\right] d\theta$$

$$\int_0^{2\pi} P(z,\theta) d\theta = 1$$

$$= \int_{I_\delta} + \int_{[0,2\pi] - I_\delta}$$

1. $\left|\int_{I_\delta}\right| \le \int_0^{2\pi} P(z,\theta)\,\left|U(\theta)-U(\theta_0)\right| d\theta$

$$\le \frac{4(1-|z|)}{\pi\delta^2}\cdot 2M\,(2\pi)$$

$M = \operatorname{Max}|U|$

$\to 0$ as $|z|\to 1$ inside $D_{\delta/2}(e^{i\theta_0})$.

2. $\left|\int_{[0,2\pi]-I_\delta}\right| \le \int_{[0,2\pi]-I_\delta} P(z,\theta)\,\left|U(\theta)-U(\theta_0)\right| d\theta$

$P(z,\theta)>0$

Let $\varepsilon>0$. $\exists\,\delta>0$ such that $|U(\theta)-U(\theta_0)|<\frac{\varepsilon}{2}$ if $|\theta-\theta_0|<\delta$.

continue: $\le \frac{\varepsilon}{2}\int_{[0,2\pi]-I_\delta} P(z,\theta)\,d\theta$

$$\le \frac{\varepsilon}{2}\underbrace{\int_{[0,2\pi]} P(z,\theta)\,d\theta}_{=1!} = \varepsilon/2$$

Done! $u(z)\to U(\theta_0)$ as $z\to e^{i\theta_0}$, $z\in D_1(0)$.

Poisson formula for $D_R(z_0)$:

Compose with $f(z) = R(z+z_0)$ to get

$$u(z_0 + re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)\, u(z_0+Re^{i\psi})}{R^2 - 2rR\cos(\theta-\psi)+r^2}\, d\psi.$$

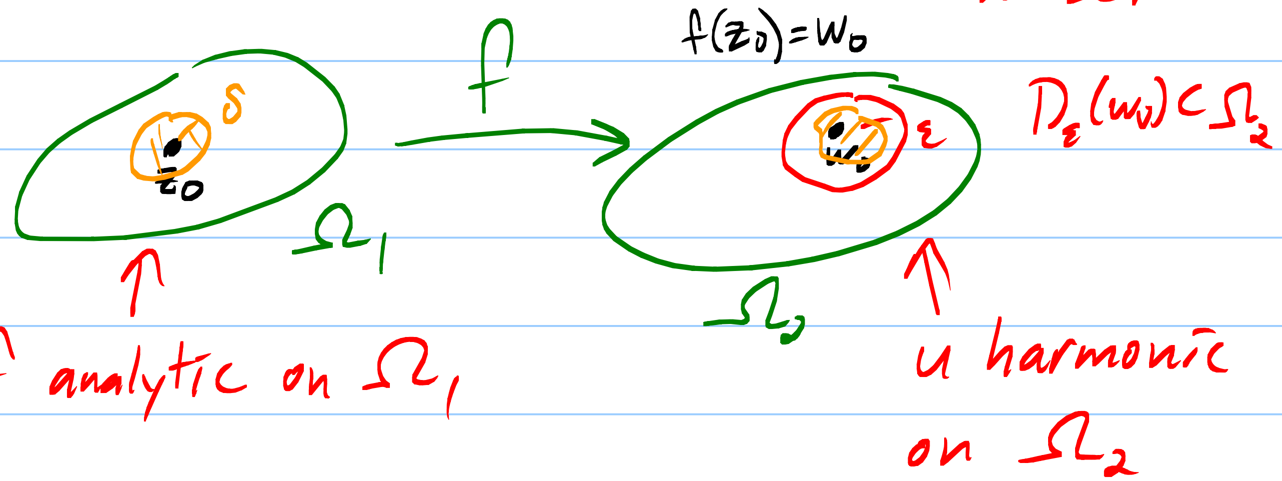
Note: $r=0$ gives ave property.

Lemma: u_n harmonic and $u_n \to u$ uniformly on compacts $\Rightarrow$ u harmonic.

Pf:
$$\begin{array}{ccc} u_n(z) & = & \int_0^{2\pi} P(z,\theta)\, u_n(e^{i\theta})\, d\theta \\ \downarrow & & \downarrow \\ u(z) & = & \int_0^{2\pi} P(z,\theta)\, u(e^{i\theta})\, d\theta \end{array}$$

$\leftarrow$ Harmonic by step 1 in Schwarz.

Fact:



Then $u \circ f$ is harmonic on Ω_1.

$u = \text{Re}\, F$ on $D_\varepsilon(w_0)$. ← F analytic only on $D_\varepsilon(w_0)$

So $u \circ f = \text{Re}[F \circ f]$ on $D_\delta(z_0)$. ✓