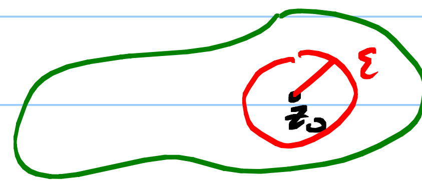
<u>Fact</u>: A continuous fcn that satisfies the averaging property is <u>harmonic</u>.

<u>Weak averaging property (WAP)</u>: A continuous function u on a domain Ω satisfies the W.A.P. if, given a point $z_0 \in \Omega$, there is an $\varepsilon > 0$ such that

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\,d\theta \quad \text{for } r$$

with $0 < r < \varepsilon$.



<u>Bigger fact</u>: WAP $\Rightarrow$ harmonic.

<u>Pf</u>: May assume u cont. on $\overline{D_1(0)}$ and satisfies WAP in $D_1(0)$. Define

$$\tilde{u}(z) = \int_0^{2\pi} P(z,\theta)\, u(e^{i\theta})\,d\theta.$$

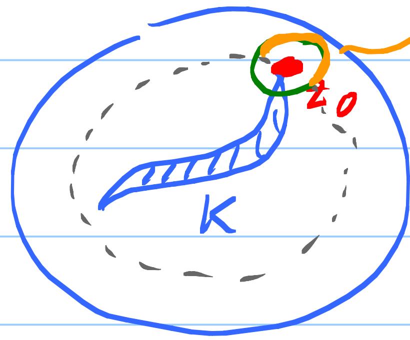
Know $\tilde{u}$ is cont on $\overline{D_1(0)}$, and harmonic on $D_1(0)$, and $\tilde{u}(e^{i\theta}) = u(e^{i\theta})$, $0 \le \theta \le 2\pi$. Harmonic fcns satisfy the WAP. Let $U = \tilde{u} - u$. Note that $U \equiv 0$ on $C_1(0)$ <u>and</u> U satisfies the WAP. Let $M = \operatorname{Max}\{U(z) : z \in \overline{D_1(0)}\}$.

Suppose that $M > 0$. Let

$K = \{ z \in D_1(0) : \tilde{U}(z) = M \}$.

$< M$ on arc!

K closed, bounded, so compact in $D_1(0)$.



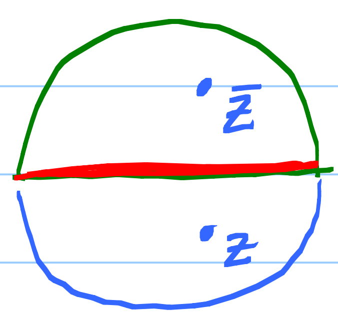
Ave prop violated at z_0.

Let z_0 be a point $z \in K$ such that $|z|$ is max.

$\lightning$. $\Rightarrow$ $M \leq 0$. Repeat argument for $u - \tilde{u}$ in place of $\tilde{u} - u$. $u \equiv \tilde{u} \leftarrow$ harmonic. ✓

Reflection Principal for harmonic fcns:

Suppose u is continuous on $D_1(0) \cap \{ \operatorname{Im} z \geq 0 \}$, and harmonic on $D_1(0) \cap \{ \operatorname{Im} z > 0 \}$, and $u(z) = 0$ for $z \in \mathbb{R}$ with $-1 < z < 1$.

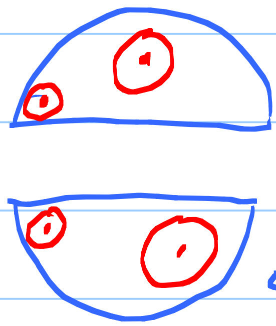


Define

$$\tilde{U}(z) = \begin{cases} u(z) & z \in D_1(0),\ \operatorname{Im} z > 0 \\ 0 & D_1(0) \cap \mathbb{R} \\ -u(\bar{z}) & z \in D_1(0),\ \operatorname{Im} z < 0 \end{cases}$$

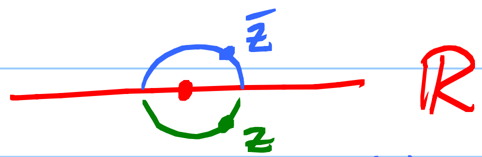
The $\tilde{U}$ is harmonic on $D_1(0)$.

Pf: Claim: $\tilde{U}$ satisfies the WAP.



$\leftarrow$ u harmonic

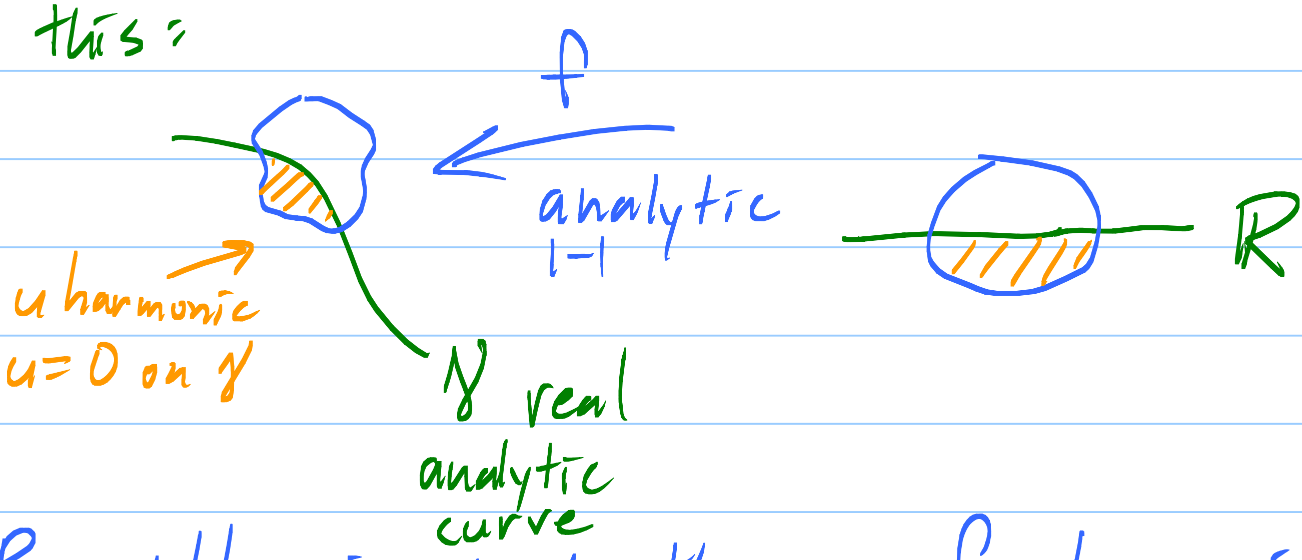
$\leftarrow$ $-u(x,-y)$ is harmonic here.



$u(z) = -u(\bar{z})$ Top cancels the bottom!

WAP $\Rightarrow$ $\tilde{U}$ is harmonic.

Remark: Schwarz reflection gets used like this:



Removable singularity theorem for harmonic fcns

Suppose u is harmonic on $D_r(z_0) - \{z_0\}$ and $|u| \le M$ there. Then u can be given a value at z_0 to make it harmonic on $D_r(z_0)$.

Pf: May assume $r=1$, $z_0=0$, u continuous on $\overline{D_1(0)}$, harmonic on $D_1(0)$.

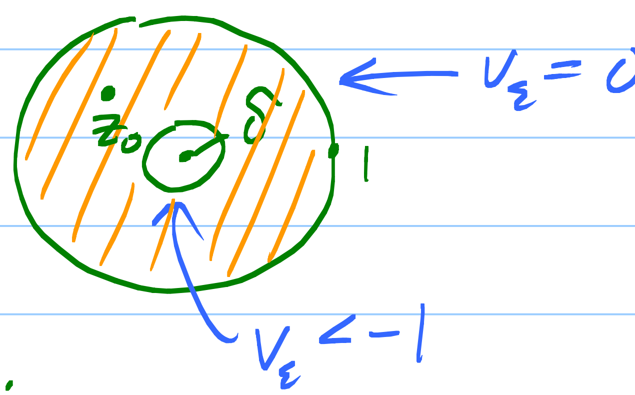
Define $\tilde{u}(z) = \int_0^{2\pi} P(z,\theta)\, u(e^{i\theta})\, d\theta$.

Let $\varepsilon > 0$. Define $V_\varepsilon(z) = u(z) - \tilde{u}(z) + \varepsilon \ln|z|$

Note that
1) V_ε harmonic on $D_1(0) - \{0\}$,
2) V_ε continuous on $\overline{D_1(0)} - \{0\}$,
3) $V_\varepsilon \equiv 0$ on $C_1(0)$, and
4) $\lim_{|z| \to 0} V_\varepsilon(z) = -\infty$. ← Need $|u| \le M$ for this.

Claim: Max Princ $\Rightarrow$ $V_\varepsilon \le 0$ on $D_1(0) - \{0\}$.

Why: Pick $z_0 \in D_1(0) - \{0\}$. Since $V_\varepsilon(z) \to -\infty$ as $z \to 0$, $\exists$ $\delta < |z_0|$ such that $V_\varepsilon(z) < -1$ if $|z| < \delta$, $z \ne 0$.



Max Princ for harmonic fcns $\Rightarrow$ Max V_ε falls on boundary. So $V_\varepsilon(z_0) \le 0$.

What? $v_\varepsilon = u - \tilde{u} + \varepsilon \ln|z| \le 0$

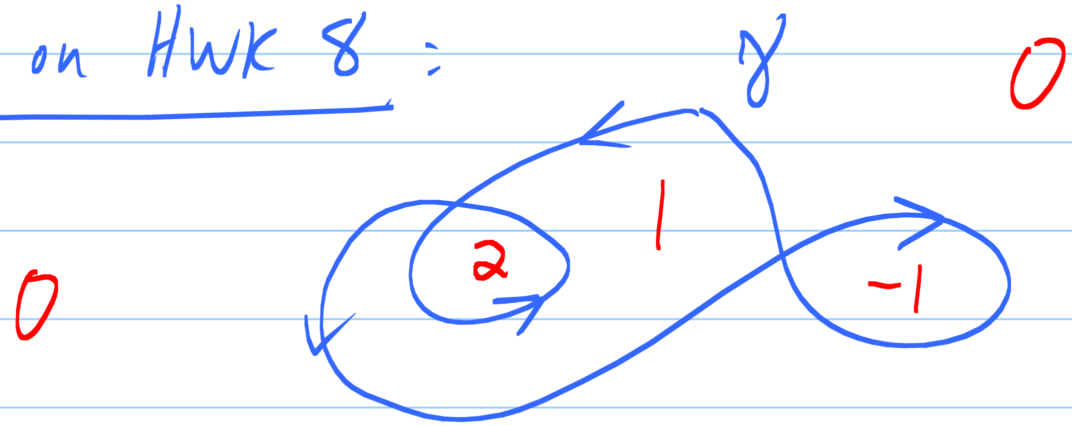
Let $\varepsilon \searrow 0$. See $u - \tilde{u} \le 0$.

Repeat, using $v_\varepsilon = \tilde{u} - u + \varepsilon \ln|z|$.

Get $\tilde{u} - u \le 0$. So $u \equiv \tilde{u}$ ← harmonic at 0.

Define $u(0) = \tilde{u}(0)$.

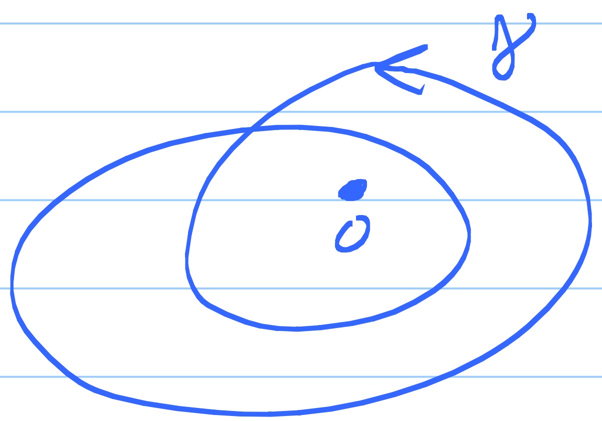
<u>Prob 1 on HWK 8</u>:



$$\mathrm{Ind}_\gamma(z) = \frac{1}{2\pi i} \int_\gamma \frac{1}{w - z}\, dw$$

= "winding number"

= # times curve goes around z.



f analytic on $\mathbb{C} - \{0\}$.

$$\int_\gamma f\, dz = \int_\gamma \begin{pmatrix} \text{Laurent} \\ \text{expansion} \end{pmatrix} dz$$

$$= \text{zero} + \int_\gamma \frac{A_{-1}}{z}\, dz \quad + \text{zero}$$

$$= 2\pi i \cdot \operatorname{Res}_0 f \cdot \operatorname{Ind}_\gamma (0)$$

Fact: $\operatorname{Ind}_\gamma$ is an integer.

Try $\quad F(z) = \exp\left(\frac{1}{3} \int_{\gamma_a^z} \frac{f'(w)}{f(w)}\, dw \right).$