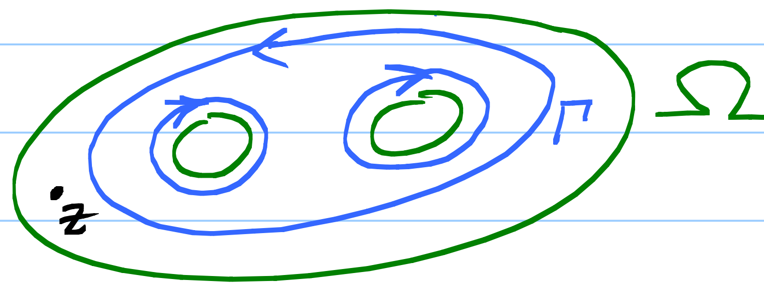
Back to proof of General Cauchy Thm

$$g(z,w) = \begin{cases} \dfrac{f(z)-f(w)}{z-w} & z \neq w \\ f'(z) & z = w \end{cases}$$ ← continuous on $\Omega \times \Omega$

$$h(z) = \frac{1}{2\pi i} \int_\Gamma g(z,w)\, dw$$ ← h analytic on Ω



Note that $$h(z) = \underbrace{\frac{1}{2\pi i} f(z) \int_\Gamma \frac{1}{z-w}\, dw}_{=0 \text{ if } \mathrm{Ind}_\Gamma(z) = 0} - \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{z-w}\, dw$$

Let $\tilde{\Omega} = \{ z \in \mathbb{C} - \mathrm{tr}(\gamma) : \mathrm{Ind}_\Gamma(z) = 0 \}$. ← includes $|z| > R$ for some R.

Notice that $\mathbb{C} = \Omega \cup \tilde{\Omega}$ ← open.

On $\tilde{\Omega}$, define $\tilde{h}(z) = -\dfrac{1}{2\pi i} \displaystyle\int_\Gamma \frac{f(w)}{z-w}\, dw$.

<u>Claim</u>:
$$H(z) = \begin{cases} h(z) & z \in \Omega \\ \tilde{h}(z) & z \in \tilde{\Omega} - \Omega \end{cases}$$ is <u>entire</u>.

Pick $z_0 \in \mathbb{C}$. Cases: $z_0 \in \Omega$ done

$z_0 \in \tilde{\Omega}$ easy

$z_0 \in \tilde{\Omega} \cap \Omega$ easy

Next: $H(z) \to 0$ as $z \to \infty$.

Because, outside $D_R(0)$, $\mathrm{Ind}_\Gamma \equiv 0$.

So for $|z| > R$, $H(z) = \frac{-1}{2\pi i} \int_\Gamma \frac{f(w)}{z-w}\, dw$

and Basic Estimate $\Rightarrow H \to 0$ as $z \to \infty$.

Liouville's $\Rightarrow H \equiv$ const, and so $H \equiv 0$.

Cauchy Integral follows:

$$H(z) = \frac{1}{2\pi i} f(z) \int_\Gamma \frac{1}{z-w}\, dw - \frac{1}{2\pi i} \int \frac{f(w)}{z-w}\, dw = 0$$

Standard Trick: Pick $z_0 \in \Omega - tr(\Gamma)$.

Given analytic f on Ω, let $F(z) = (z-z_0) f(z)$

Apply Cauchy Integral Formula to F at z_0.

$$\frac{1}{2\pi i} \int_\Gamma \frac{F(w) = (w-z_0)f(w)}{w-z_0}\, dw = \underbrace{F(z_0)}_{0} \mathrm{Ind}_\Gamma(z_0)$$

Aha! $\int_\Gamma f(w)\,dw = 0$.

HWK 8 : 2. $a_j \in \mathbb{C}$, $A = \{a_j\}_{j=1}^N$.

$F = f - \sum_{j=1}^N \frac{c_j}{z-a_j} = g' \leftarrow$ on $\mathbb{C} - A$

<u>Facts</u>: 1) $F = g' \iff \int_\gamma F\,dz = 0$ for all closed γ in $\Omega = \mathbb{C} - A$.

2) Let γ be a closed curve in Ω.

Residue Theorem: $\int_\gamma f\,dz = 2\pi i \sum_{j=1}^N \text{Ind}_\gamma(a_j)\,\text{Res}_{a_j} f$

<u>Claim</u>: Gen^l Cauchy $\Rightarrow$ Res. Thm.

<u>Why</u>: Let $n_j = \text{Ind}_\gamma(a_j)$.

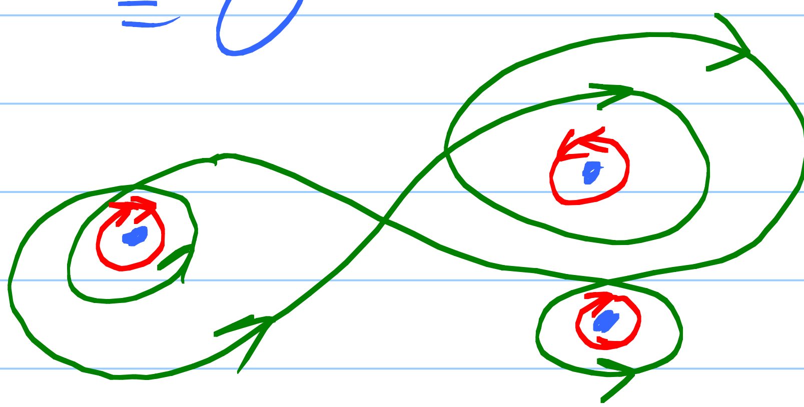
$-n_j C_j$: $z(t) = a_j + \varepsilon_j e^{i(-n_j)\theta}$, $0 \le \theta \le 2\pi$

Note: $\text{Ind}_{-n_j C_j}(a_j) = -n_j$.

Big idea: $\Omega = \mathbb{C} - A$. $\Gamma = \gamma \cup \left(\bigcup_{j=1}^N -n_j C_j\right)$

Make ε_j's so small that $\overline{D_{\varepsilon_j}(a_j)} \cap \overline{D_{\varepsilon_k}(a_k)} = \emptyset$ if $j \neq k$. Then

$$\mathrm{Ind}_\Gamma(a_j) = \mathrm{Ind}_\gamma(a_j) + 0 + \cdots + 0 + (-n_j) + 0 + \cdots + 0$$
$$= 0$$



Gen'l Cauchy $\Rightarrow$ $\displaystyle\int_\Gamma f\,dz = 0$

$$0 = \int_\gamma f\,dz + \sum_{j=1}^{N} \underbrace{\int_{-n_j C_j} f\,dz}$$

$$\underbrace{-n_j}_{\mathrm{Ind}_\gamma(a_j)} \underbrace{\int_{C_j} f\,dz}_{= 2\pi i \,\mathrm{Res}_{a_j} f}$$

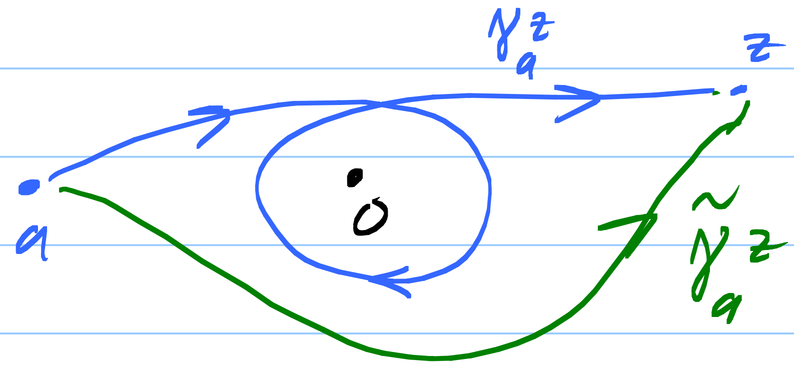
Aha! Let $c_j = \mathrm{Res}_{a_j} f$. Apply Residue Thm to

$$\int_\gamma \left(f - \sum_{j=1}^{N} \frac{c_j}{z - a_j} \right) dz = 0$$

Prob 1: Try

$$F(z) = \exp\left(\frac{1}{3}\int_{\gamma_a^z} \frac{f'(w)}{f(w)}\,dw\right)$$

Well defined? $\gamma = \gamma_a^z \cup (-\tilde{\gamma}_a^z)$ ← closed

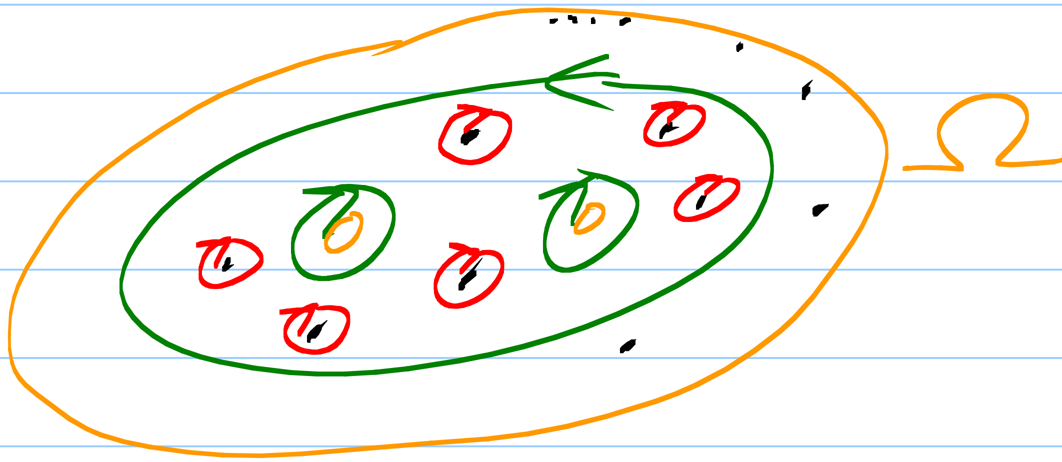


Hint $\dfrac{F(z)}{\tilde{F}(z)} = \dfrac{\exp(\ell\ell\ell)}{\exp(\ell\ell\ell)} = \exp\Big(\underbrace{\ell\ell\ell - \ell\ell\ell}_{\text{closed curve!}}\Big)$

Residue Thm: $\displaystyle\int_C \frac{f'}{f}\,dz = 2\pi i\, \mathrm{Res}_0 \frac{f'}{f}$

General Residue Theorem: Suppose Γ is a cycle in a domain Ω such that $\mathrm{Ind}_\Gamma(a) = 0$ for all $a \in \mathbb{C} - \Omega$, and suppose A is a discrete set in Ω. If f is analytic on $\Omega - A$, then $\displaystyle\int_\Gamma f\,dz = 2\pi i \sum_{a\in A} \mathrm{Ind}_\Gamma(a)\,\mathrm{Res}_a f$. ← finite sum.

<u>Note</u>: $\mathrm{Ind}_\Gamma(a) \neq 0$ for only finitely many $a \in A$.



Let $K = \{z \in \mathbb{C} : \mathrm{Ind}_\Gamma(z) \neq 0\} \cup tr(\Gamma)$.

closed and bounded, so compact.

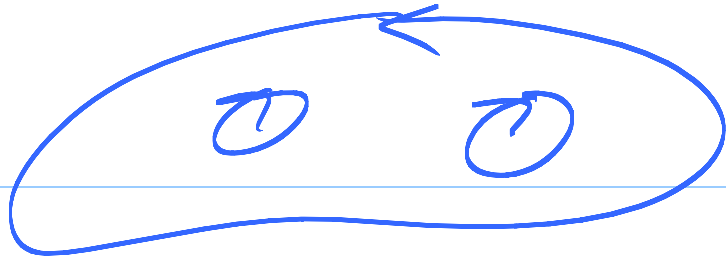
$\Gamma \sim 0 \Rightarrow K \subset \Omega$. If $K \cap A$ infinite, it would have a limit pt in $K \subset \Omega$. $\lightning$.

Apply Gen^l Cauchy to $\tilde{\Omega} = \Omega - A$

$$\Gamma = \gamma \cup \left(\tilde{\bigcup_{j=1}} -n_j C_j \right) \leftarrow \Gamma \sim 0 \text{ in } \tilde{\Omega}$$

<u>Remark</u>: Simple Cycle: $\mathrm{Ind} = 1$ "inside"

$\mathrm{Ind} = 0$ "outside"



$$\int_\Gamma f\,dz = 2\pi i \sum_{a\in A} \mathrm{Res}_a f$$

A = Singular points "inside" Γ.

<u>Argument Princ</u>: $\displaystyle \frac{1}{2\pi i}\int_\Gamma \frac{f'}{f}\,dz = \left(\begin{array}{l}\#\text{ zeroes} \\ \text{"inside" } \Gamma \\ \text{counted with mult}\end{array}\right)$

$$\left(= \left(\#\text{zeroes}\right) - \left(\#\text{poles}\right) \text{ allowing poles}\right)$$