<u>Mittag-Leffler</u>: Given a seq of discrete pts $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{C}$ (i.e., $a_n \to \infty$ as $n \to \infty$). [Note: a_n are distinct too.] Given principal parts

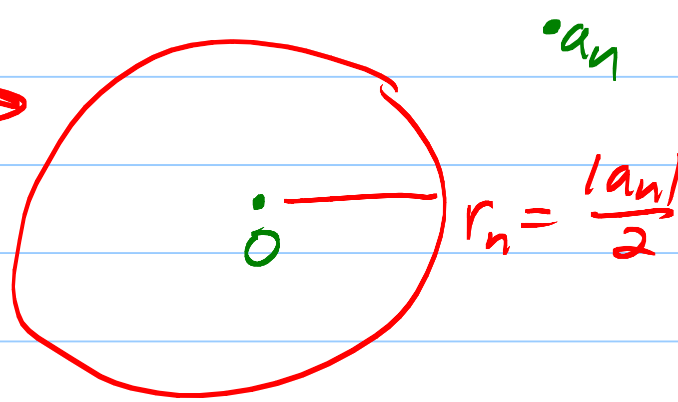
$$R_n(z) = \frac{A_{nN_n}}{(z-a_n)^{N_n}} + \cdots + \frac{A_{n1}}{(z-a_n)}.$$

There exists an analytic fcn f on $\mathbb{C} - \{a_n\}_{n=1}^{\infty}$ with poles at each a_n with p.p. R_n.

<u>Note</u>: f is called <u>meromorphic</u>.

<u>Pf</u>: First try: $f = \sum_{n=1}^{\infty} R_n(z)$

approximate $R_n(z)$ by Taylor sum here.



Get polys $p_n(z)$ such that $|R_n(z) - p_n(z)| < \frac{1}{2^n}$

<u>Second try</u>: $f(z) = \sum_{n=1}^{\infty} \left(R_n(z) - p_n(z)\right)$

Claim: f solves the M-L Prob.

Consider f on $D_R(0)$. Since $a_n \to \infty$, $\exists N$ such $\frac{|a_n|}{2} > R$ if $n > N$.

$$f = \underbrace{\sum_{n=1}^{N} \left[R_n(z) - p_n(z)\right]}_{\text{Rational fcn with prescribed poles inside } D_R(0).} + \sum_{n=N+1}^{\infty} \underbrace{\left[R_n(z) - p_n(z)\right]}_{|\cdots| < \frac{1}{2^n} \text{ on } D_R(0)}$$

So this Σ converges to analytic fcn on $D_R(0)$

Near a_k: $f = R_k +$ (analytic near a_k).

Remark: M-L Thm is true on any domain in $\mathbb{C}$.

Weierstraß Thm: Given a discrete set of points $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{C}$ and positive integers m_n, there is an entire fcn with zeroes of order m_n at a_n for each n and no other zeroes.

One proof: Use infinite products.

Or: M-L Thm gives us a meromorphic

fcn $F(z)$ with simple poles at a_n with p.p. $\frac{m_n}{z-a_n}$ for each n.

Think: $f = e^{\log f} = e^{\int_{\gamma_a^z} \frac{f'}{f}\,dw}$

$= \text{near } a_n = e^{\int_{\gamma_a^z} \underbrace{\frac{m_n}{w-a_n}} + (\text{analytic})\,dw}$

Idea: Try $f(z) = \exp\left(\int_{\gamma_a^z} F(w)\,dw \right)$

where γ_a^z is a curve starting from a fixed a and going to z, avoiding a_n's.

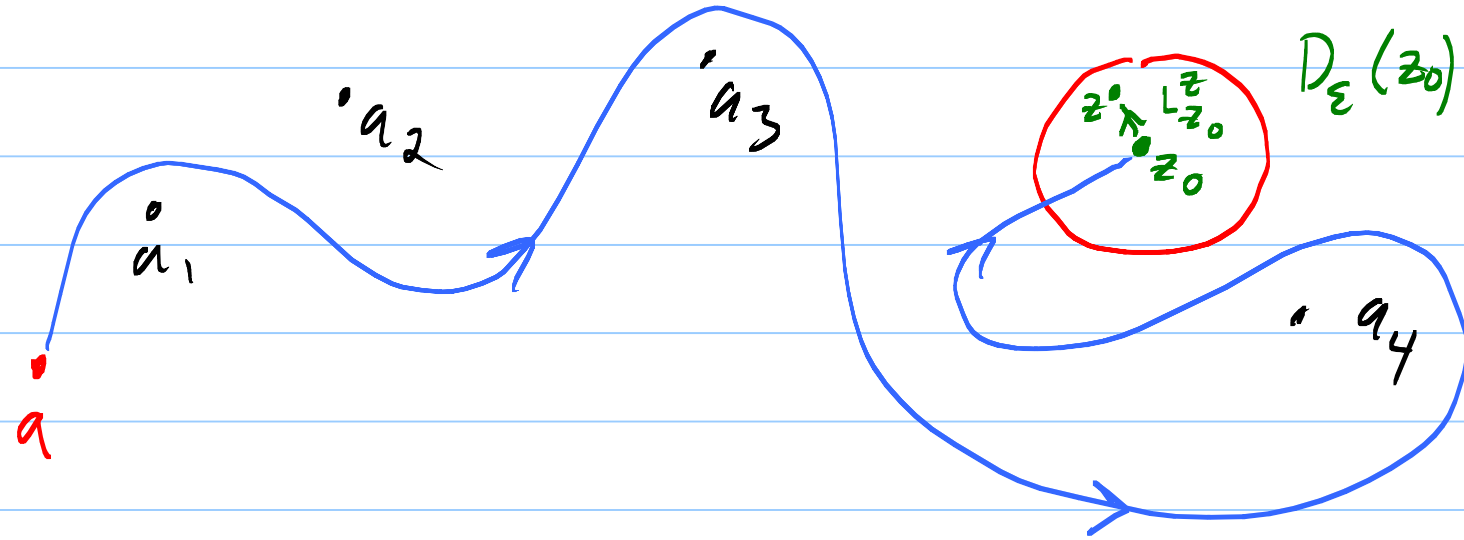
Step 1: f is well defined.

$$\frac{f(z)}{\tilde{f}(z)} = \exp\left(\underbrace{\int_{\overbrace{\gamma_a^z \cup (-\tilde{\gamma}_a^z)}^{\Gamma}} F(w)\,dw}_{2\pi i \sum \underbrace{\mathrm{Ind}_\Gamma(a_n)}_{\text{integer}} \underbrace{\mathrm{Res}_{a_n} F}_{m_n}} \right)$$

$= 2\pi i\,(\text{integer})$.

So $\frac{f}{\tilde{f}} = 1$.

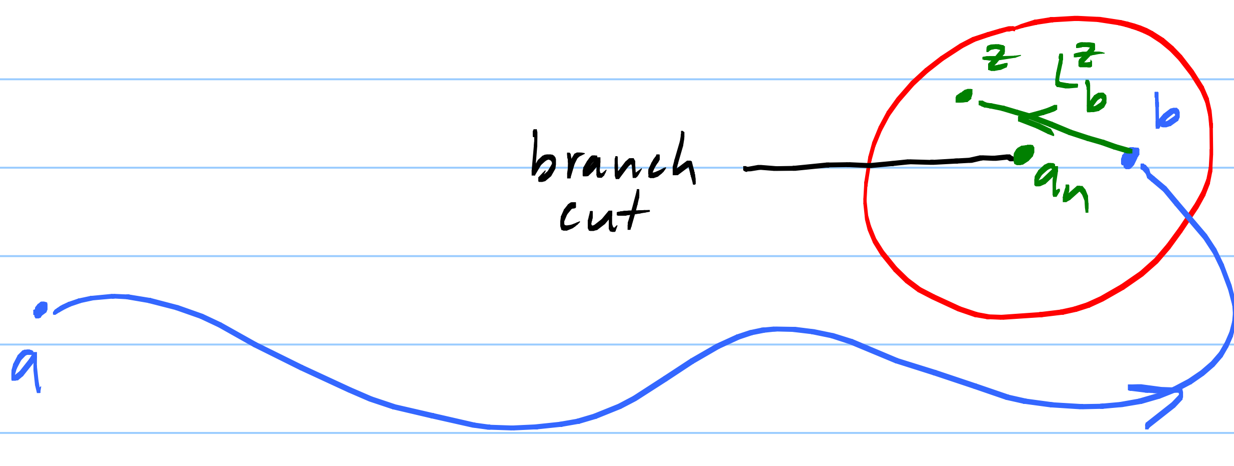
Step 2: f is analytic on $\mathbb{C} - \{a_n\}_{n=1}^{\infty}$



on $D_\varepsilon(z_0)$, $f(z) = \exp\Big(\underset{\substack{\uparrow \\ \text{const} = \int_{\gamma_a^{z_0}}}}{c} + \underbrace{\int_{L_{z_0}^{z}} F(w)\,dw}_{\text{analytic!}} \Big)$

[Hmmm. $f'(z) = f(z)F(z)$.]

Step 3: a_n is a removable sing.



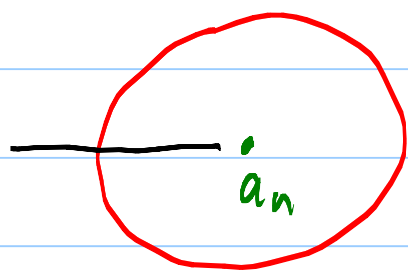
$$f(z) = \exp\Big(\underset{\uparrow}{C} + \underbrace{\int_{L_b^z} F\,dw}_{\int_{L_b^z} \underbrace{\frac{m_n}{w-a_n}}_{m_n \operatorname{Log}(w-a_n)\big|_{z_0}^{z}} + \underset{\uparrow}{H}\,dw} \Big)$$

$\int_{\gamma_a^b} F\,dw$

analytic

$$= \underbrace{\exp\big(\operatorname{Log}(z-a_n)\big)^{m_n}}_{(z-a_n)^{m_n}} \; \underbrace{\exp\big(\text{Nice and analytic}\big)}_{h(z)}$$

$\uparrow$ analytic. $h(a_n) \neq 0$.



Step 4: Aha! f is analytic on $D_\varepsilon(a_n) - (\text{branch cut})$, continuous up to cut. So it extends to be analytic on whole $D_\varepsilon(a_n)$ and $f(z) = (z-a_n)^{m_n} \underbrace{h(z)}_{\text{analytic on } D_\varepsilon(a_n).}$ there

Done!

<u>Remark</u>: Weirstraß Thm true on general domains.

Read Stein: Radó's Thm
Infinite products.
Runge's Thm.