Last year final. #4.

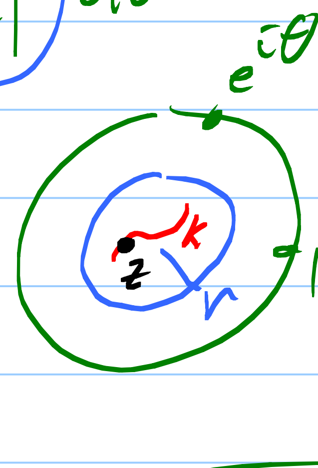
Montel's Thm.: f_n unif bounded on compact subsets. Then $\exists$ subseq that converges unif on compacts.

Suppose $K \subset\subset D_1(0)$. $\exists$ r with $0<r<1$ such that $K \subset D_r(0)$. For $z \in D_r(0)$,

$$|f_n(z)| = \left| \frac{1}{2\pi i} \int_{C_1} \frac{f_n(w)}{w-z}\, dw \right| = \left| \lim_{\rho \nearrow 1} \frac{1}{2\pi i} \int_{C_\rho} \right|$$

$$= \frac{1}{2\pi} \left| \int_0^{2\pi} \frac{f_n(e^{i\theta})}{e^{i\theta}-z}\, i e^{i\theta}\, d\theta \right|$$

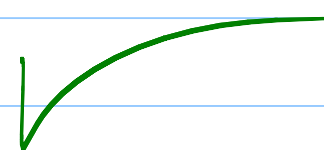
$$\leq \frac{1}{2\pi} \int_0^{2\pi} |f_n(e^{i\theta})| \frac{1}{|e^{i\theta}-z|}\, d\theta$$



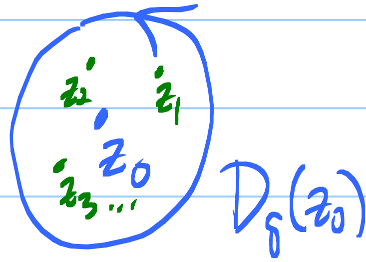
Aha!

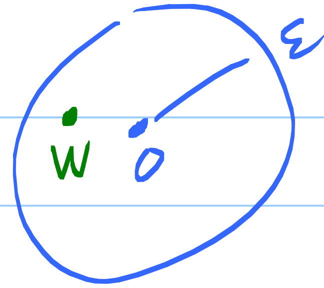
$$\leq \frac{1}{1-r}$$

$$\leq \frac{1}{2\pi(1-r)} \underbrace{\int_0^{2\pi} |f_n(e^{i\theta})|\, d\theta}_{\leq 1}$$



#6.





Rouché's: $f(z)$, $g(z) = f(z) - w$

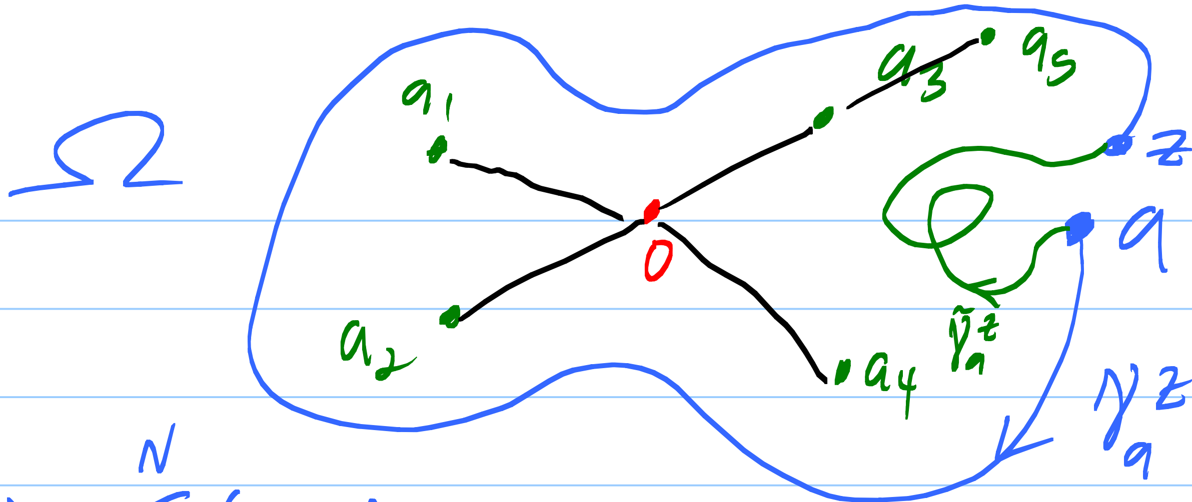
$$|f(z) - g(z)| = |w| \underset{\text{want}}{<} \overset{\varepsilon!}{m} = \min_{C_\delta(z_0)} |f(z)| \leq |f(z)|$$

Shrink δ so that z_0 is the only zero of f in $\overline{D_\delta(z_0)}$. Aha! Take $\varepsilon = m = \min\limits_{C_\delta(z_0)} |f|$.

Rouché $\Rightarrow$ $f(z) = w$ has N roots (counted with multiplicity). Finally, show that roots are each mult <u>one</u> to get $\underline{\underline{N}}$ distinct roots.

Shrink δ again so that z_0 is the <u>only</u> possible zero of f' in $\overline{D_\delta(z_0)}$. All N zeroes are simple now.

5



$$F(z) = \prod_{n=1}^{N} (z - a_n)$$

Try $\quad f(z) = \exp\left(\frac{1}{N} \int_{\gamma_a^z} \frac{F'(w)}{F(w)}\, dw \right)$

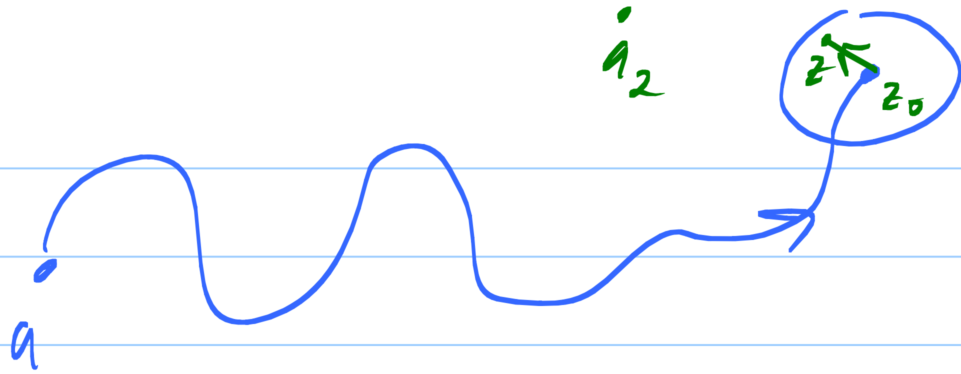
Well defined: $\quad \frac{f(z)}{\tilde{f}(z)} = \exp\left(\underbrace{\frac{1}{N} \int_{\overbrace{\gamma_a^z \cup (-\tilde{\gamma}_a^z)}^{\Gamma}} \frac{F'}{F}\, dw}_{2\pi i \sum_{n=1}^{N} \operatorname{Ind}_\Gamma(a_n) \underbrace{\operatorname{Res}_{a_n} \frac{F'}{F}}_{1}} \right)$

Residue Thm on $\mathbb{C}$ ← simply conn.

Aha! All a_n's in same component of $\mathbb{C} - \operatorname{tr}(\Gamma)$. So $\operatorname{Ind}_\Gamma(a_n) = \text{same } m$ for all n.

So $\quad \frac{f(z)}{\tilde{f}(z)} = \exp\left(\frac{1}{N} \cdot 2\pi i\, N \cdot m \cdot 1 \right) = 1$

Step 2: $f(z)$ is analytic:



Step 3: $f(z)^N = F$?

$$f(z) = \exp\left(\frac{1}{N}\int_{\gamma_a^z} \frac{F'}{F}\,dw\right)$$

$$f'(z) = \underbrace{\exp\left(\ \ldots\ \right)}_{f}\frac{1}{N}\frac{F'}{F}$$

$$\boxed{\frac{f'}{f} = \frac{1}{N}\frac{F'}{F}}$$

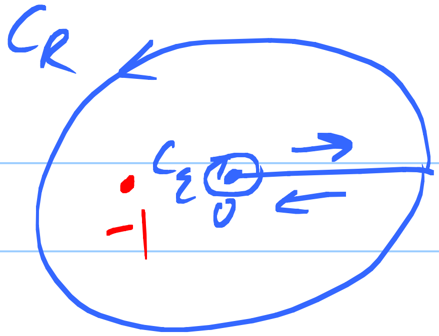
$\leftarrow$ this shows that $\dfrac{d}{dz}\left[\dfrac{F(z)}{f(z)^N}\right] \equiv 0$.

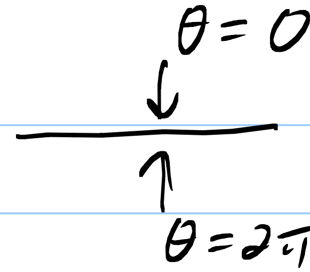
So $F(z) = c\,f(z)^N$ on Ω.

$$= \Big[\underbrace{c^{1/N} f(z)}\Big]^N$$

this is it!

#2





$$z^{1/5} = e^{\frac{1}{5}\log z} = e^{\frac{1}{5}(\ln|z| + i\theta)}$$

$\theta \in \arg z$

$0 < \theta < 2\pi$

$$|z^{1/5}| = |z|^{1/5}.$$

$$\left| \int_{C_R} \frac{1}{z^{1/5}(z+1)}\, dz \right| \le \frac{1}{R^{1/5}} \cdot \frac{1}{R-1} (2\pi R)$$

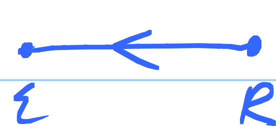
$$\to 0 \quad \text{as } R \to \infty,$$

$$\left| \int_{C_\varepsilon} \right| \qquad \frac{1}{\varepsilon^{1/5}} \cdot \frac{1}{1-\varepsilon} \cdot (2\pi\varepsilon)$$

$$\to 0 \quad \text{as } \varepsilon \to 0.$$

$\leftarrow$ want!

L:



$-L: \ z(t) = t, \quad \varepsilon \le t \le R.$

$$\int_L = -\int_{-L} = -\int_\varepsilon^R \frac{1}{e^{\frac{1}{5}(\ln t + i2\pi)}(t+1)}\, 1 \cdot dt$$

$$= -\frac{1}{e^{i2\pi/5}}\left(\int \text{we want}\right)$$

Residue Thm. $\sum \int = 2\pi i \operatorname{Res}_{-1} \frac{1}{z^{1/5}(z+1)}$

$$= 2\pi i \frac{1}{(-1)^{1/5}}$$

$$= 2\pi i \frac{1}{e^{\frac{1}{5}(\text{Ln}|-1| + i\pi)}}$$

$$= \frac{2\pi i}{e^{i\pi/5}}$$