Lecture 10 Zeroes of analytic fcns

Lemma: Suppose f analytic on $D_R(z_0)$ with $f(z_0)=0$.

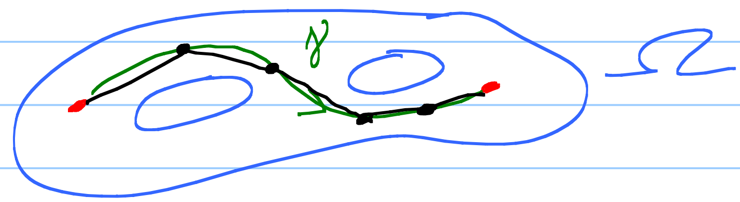
Then, either A) $f \equiv 0$ on $D_R(z_0)$, or

B) the zero of f at z_0 is isolated (meaning, $\exists\, \delta$ with $0<\delta<R$ such that $f(z)\neq 0$ for $z\in D_\delta(z_0)-\{z_0\}$.)

Connected Open Sets in $\mathbb{C}$ (Domains)

Def^n 1: Suppose $\Omega\subset\mathbb{C}$ is open. Ω is path connected if any two points in Ω can be joined by a piecewise $\mathcal{C}^1$-smooth in Ω.

Remark: Topology def^n uses paths merely continuous.



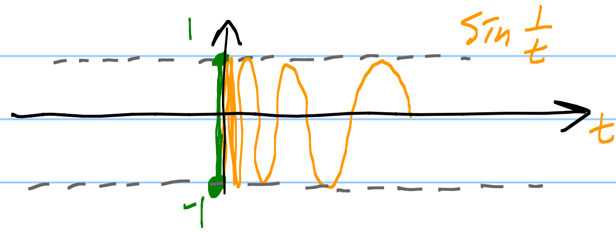
γ continuous

Polygonal path is piecewise $\mathcal{C}^1$-smooth

Def^n 2: If U_1 and U_2 are disjoint open sets such that $\Omega = U_1\cup U_2$, then either $\begin{cases} U_1=\Omega \\ U_2=\emptyset\end{cases}$ or $\begin{cases} U_2=\Omega \\ U_1=\emptyset\end{cases}$.

Fact: Def^n 1 and 2 are equivalent for open $\Omega\subset\mathbb{C}$.

EX:



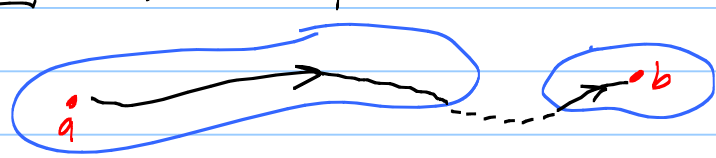
$[$ union $\sin(1/t)$ graph

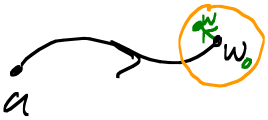
Def 2: yes.

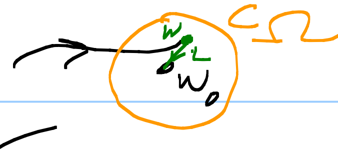
Def 1: no

Pf of Fact: 1) Ω open connected #2. Let $a\in\Omega$.

$U_1=\{w\in\Omega$ such that $\gamma_a^w\subset\Omega.\}$

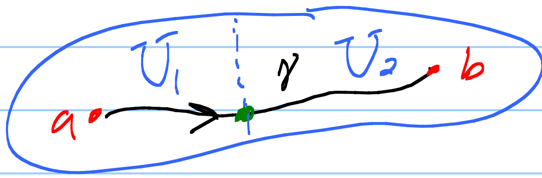


U_1 is open  $\subset \Omega$. Let $U_2 = \Omega - U_1$.

Claim: U_2 open  Can't connect to w_0. Aha! Can't connect to w in small disc.

$U_1 \neq \emptyset$, so $U_1 = \Omega$. ✓

2) Suppose Ω path connected. Suppose $\Omega = U_1 \cup U_2$ open, disjoint



Let $t_\infty = \sup\{t : z(t) \in U_1\}$.

Where is $z(t_\infty)$? Not in either!

MATH 530: Analytic fcns on a domain.

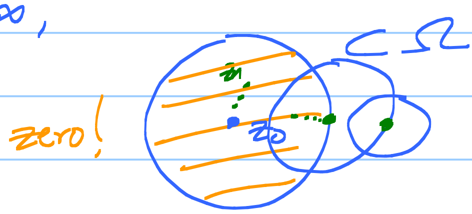
Theorem: If f analytic on a domain and $f' \equiv 0$, then f is constant.

Pf: γ_a^z in Ω. $0 = \int_{\gamma_a^z} \underbrace{f'\, dw}_{=0} = f(z) - f(a)$

Theorem: (Identity theorem) Suppose f analytic on domain Ω. Let $Z_f = \{z : f(z) = 0,\ z \in \Omega\}$. Then, either

1) $f \equiv 0$ on Ω
2) Z_f has no limit points in Ω.

Note: $z_0 \in \Omega$ is a limit pt of Z_f means $\exists$ seq $\{z_n\}_{n=1}^\infty$ with $z_n \in Z_f$ and $z_n \neq z_0$ for each n and $z_n \to z_0$ as $n \to \infty$.



[Remark: If z_0 is a limit pt of zeroes, $f(z_0) = 0$ too by continuity.

Pf of Thm: Let U_1 denote the set of all limit points of zeroes of f in Ω. Suppose $z_0 \in U_1$. Ω open. So $\exists$

$D_R(z_0) \subset \Omega$. (Note: $f(z_0) = 0$ too.) Lemma $\Rightarrow$ $f \equiv 0$ on $D_R(z_0)$. So $D_R(z_0) \subset U_1$. So U_1 is open.

Next: Show $U_2 = \Omega - U_1$ is open too. Suppose $w_0 \in U_2$. Then w_0 is not a limit pt of Z_f.

Case 1: $f(w_0) \neq 0$, then $\exists\ \delta > 0$ such $f(z) \neq 0$ on $D_\delta(w_0) \subset \Omega$, and we see that $D_\delta(w_0) \subset U_2$.

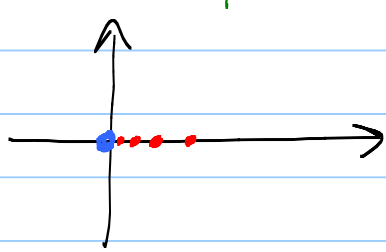
Case 2: $f(w_0) = 0$. w_0 not a limit pt of Z_f $\Rightarrow$ w_0 is an isolated zero (by Lemma). $\exists\ \delta > 0$ such that $f(z) \neq 0$ for $z \in D_\delta(w_0) - \{w_0\}$ $\subset U_2$. And so $D_\delta(w_0) \subset U_2$. U_2 is open.

Conclude $U_1 = \Omega$, $U_2 = \emptyset$ (because $z_0 \in U_1$).

So $f \equiv 0$ on Ω!

Cor: If f, g analytic on a domain and they agree on a set with a limit pt in the domain, then $f \equiv g$.

EX:



$\sin \frac{1}{z}$ zeroes at $z = \frac{1}{n\pi}$

$\Omega = \mathbb{C} - \{0\}$

OK to have limit pts of zeroes in the boundary of Ω.

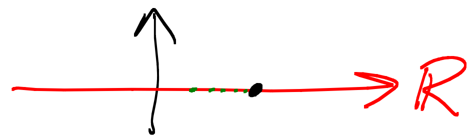
Identity Thm is way false $\mathbb{R} \to \mathbb{R}$.

$$h(t) = \begin{cases} e^{-1/t^2} \sin \frac{1}{t} & t \neq 0 \\ 0 & t = 0 \end{cases} \qquad C^\infty \text{ smooth!}$$

Zeroes pile up at $t = 0$. Also $h \neq$ Taylor series about 0.

Application: 1) e^z is the only fcn that extends e^x from $\mathbb{R}$

to $\mathbb{C}$ as an analytic fcn.



2) Trig identities hold for <u>complex</u> angles!

$$\sin 2\theta = 2 \sin\theta \cos\theta$$

$$\sin(\alpha+\beta) = \cos\alpha \sin\beta + \cos\beta \sin\alpha$$