Lecture 15 Partial fractions, Schwarz Lemma, log

Partial Fractions Suppose $P(z)$ and $Q(z)$ are polys with

$N_Q = \deg Q > \deg P = N_P$. P, Q have no common factors.

Step 1: Let a_k denote the roots of Q, $k=1,2,\ldots,M \le N_Q$.
Multiplicity of a_k is m_k. Then, near a_k:

$Q(z) = (z-a_k)^{m_k} q_k(z)$ where q_k is poly with
$q_k(a_k) \neq 0$.

Step 2: $$\frac{P(z)}{Q(z)} = \frac{1}{(z-a_k)^m} \cdot \frac{P(z)}{q_k(z)}$$ ← analytic near a_k, not zero at a_k.

So $\frac{P}{Q}$ has a pole of order m_k at a_k.

Let R_k denote the princ. part at a_k:

$$R_k(z) = \frac{A_1}{(z-a_k)^{m_k}} + \cdots + \frac{A_{m_k}}{z-a_k}$$

Recall: $\frac{P}{Q} - R_k$ has a removable sing at a_k.

Step 3: Let $f = \frac{P}{Q} - \sum_{k=1}^{M} R_k$

Near a_p: $$f = \underbrace{\left(\frac{P}{Q} - R_p\right)}_{\text{removable sing at } a_p} - \underbrace{\sum_{k\neq p} R_k}_{\text{analytic near } a_p}$$

Give f the value at a_p to make it analytic there.
f is entire!

Step 4: f is bounded!

Basic Poly Estimates:

$$a|z|^{N_P} \le |P(z)| \le A|z|^{N_P} \qquad |z| > R_P$$

$$b|z|^{N_Q} \le |Q(z)| \le B|z|^{N_Q} \qquad |z| > R_Q$$

$$\left|\frac{P(z)}{Q(z)}\right| \le \frac{A}{b}\frac{1}{|z|^{\underbrace{N_Q-N_P}_{>0}}} \qquad \text{if} \quad |z| > \mathrm{Max}(R_P, R_Q)$$

So $\frac{P}{Q} \to 0$ as $|z| \to \infty$. Aha! So do all the terms in the princ. parts. So $|f| \to 0$ as $|z| \to \infty$, and f is bnded entire.

Liouville's $\Rightarrow$ $f \equiv$ const, which must be zero.

So $\frac{P}{Q} = \sum_{k=1}^{M} R_k$ ← Partial fractions!

Freshman calculus: $\frac{x^3+7x+13}{(x-2)^2(x^2+x+1)} = \frac{A}{(x-2)^2} + \frac{B}{(x-2)} + \frac{Cx+D}{\underbrace{x^2+x+1}}$

$\frac{E}{x-r} + \frac{F}{x-\bar{r}}$

E, F complex

Schwarz Lemma: (Chapter 8, sec 2)

Suppose $f: D_1(0) \to D_1(0)$ is analytic and $f(0) = 0$.

$|f(z)| < 1$ when $|z| < 1$,

Then 1) $|f(z)| \le |z|$

2) $|f'(0)| \le 1$

Furthermore, if $|f(z)| = |z|$ for some $z \ne 0$, or if $|f'(0)| = 1$, then $f(z) \equiv \lambda z$ where λ is a const with $|\lambda| = 1$.

Pf: Define $F(z) = \begin{cases} \frac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0 \end{cases}$ $\quad \frac{f(z)}{z} = \underbrace{\frac{f(z)-f(0)}{z-0}}_{DQ \to f'(0)}$

So F is cont at $z=0$, analytic on $D_1(0) - \{0\}$.

R.R.S.T. $\Rightarrow$ F analytic on $D_1(0)$.

Pick $z_0 \in D_1(0)$. Let r be such that $|z_0| < r < 1$.

Max princ on $\overline{D_r(0)}$: $\quad |F(z_0)| \leq \max\limits_{C_r(0)} |F| = \max\limits_{|z|=r} \frac{|f(z)|}{|z|} \leq \frac{1}{r}$.

Let $r \nearrow 1$. Then $\frac{1}{r} \searrow 1$. Conclude $|F(z_0)| \leq 1$. (z_0 arbitrary!)

So 1 holds if $z \neq 0$ (and at $z=0$ too).

2 holds at $z=0$.

Furthermore: If $|f(z)| = |z|$ for $z \neq 0$, then $|F(z)| = \left|\frac{f(z)}{z}\right| = 1$ there. And $|f'(0)| = 1$, then $|F(0)| = \underline{1}$ $\leftarrow$ $|F|$ has a local $\underline{\max}$ in $D_1(0)$

Max Princ #1 $\Rightarrow$ $F \equiv c$. See $|c| = 1$.

<u>Logs and roots on a convex open set:</u>

<u>Thm</u>: Given a <u>non-vanishing</u> analytic fcn F on a convex open set Ω (later Ω = simply connected domain), there is an analytic log fcn G on Ω such that $e^G = F$ on Ω.

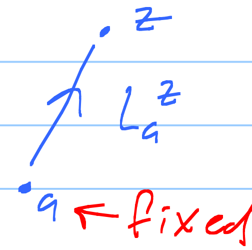
<u>Idea</u>: If we had G:

$$F = e^G \qquad (1)$$
$$F' = e^G \cdot G' \qquad (2)$$

Divide (2) by (1): $G' = \dfrac{F'}{F}$

Pf: Aha! Define

$$G(z) = \int_{L_a^z} \frac{F'(w)}{F(w)}\,dw$$



We know $G'(z) = \dfrac{F'(z)}{F(z)}$. ← $F' = F G'$

Wish $F = e^G$. Hmmm. $\dfrac{F}{e^G} = F e^{-G}$

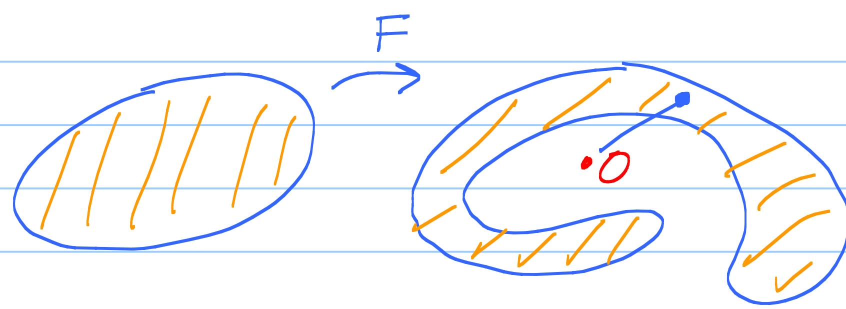
$$\frac{d}{dz}\left(\frac{F}{e^G}\right) = F' e^{-G} + F e^{-G}(-G')$$

$$= \big(\underbrace{F' - F G'}_{\equiv 0}\big) e^{-G}$$

So $\dfrac{F}{e^G} \equiv c$, a const on Ω.

$$F = c e^G = e^{\operatorname{Log} c} e^G = e^{(\underbrace{\operatorname{Log} c + G}_{\text{this is the } G \text{ we want!}})}$$

Picture:



$G = \operatorname{Ln}|F| + i\alpha$, $\alpha \in \arg F$ ← $\arg F$ has a continuous branch on Ω.

Note: N-th root: $e^G = F$

$$\big(\underbrace{e^{G/N}}_{N\text{-th root}}\big)^N = e^G = F$$

<u>Exercise:</u> Suppose H_1 and H_2 are N-th roots of non-vanishing F on a domain Ω. Show that

$$H_1 = \lambda H_2 \quad \text{where } |\lambda| = 1 \text{ and } \lambda^N = 1.$$

<u>EX:</u> $H_1^2 = F$

$H_2^2 = F$

$$H_1^2 - H_2^2 = 0$$

$$(H_1 - H_2)(H_1 + H_2) = 0$$

Hmmm. Identity Thm.