Lecture 16 Open Mapping Theorem (O.M.T.)

OMT: Non-constant analytic fcns map open sets to open sets.

Notation: $f(U) = \{ f(z) : z \in U \}$

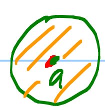
$f : \Omega \to \mathbb{C}$ $\qquad f^{-1}(U) = \{ z \in \Omega : f(z) \in U \}$

Fact: f continuous $\iff$ $f^{-1}(U)$ open whenever U is open.

Cor: If analytic f has an inverse, then the inverse is continuous. [Later: inverse must be analytic!]

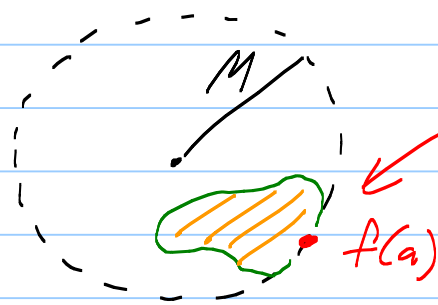
OMT is way false $\mathbb{R} \to \mathbb{R}$: $h(t) = 1 - t^2$ $\quad h : (-1,1) \to (0,1]$

Remark: OMT $\Rightarrow$ Max Princ.



$|f(a)| = M$

max on $\overline{D_r(a)}$



oops! no disc about $f(a)$ in image.

Argument Principal for a disc:

Suppose f analytic on $D_R(z_0)$ and $0 < r < R$, and f has no zeroes on $C_r(z_0)$, then

$$\frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f'(z)}{f(z)}\, dz = \left(\begin{array}{l} \#\text{ zeroes of } f \\ \text{inside } C_r(z_0) \\ \text{counted with mult.} \end{array} \right)$$

Pf: Suppose $f(a) = 0$. Then $f(z) = (z-a)^N F(z)$, F analytic, $F(a) \neq 0$.

$$\frac{f'(z) = N(z-a)^{N-1} F(z) + (z-a)^N F'(z)}{f(z) = (z-a)^N F(z)}$$

$$= \frac{f'(z)}{f(z)} = \frac{N}{(z-a)} + \frac{F'(z)}{F(z)}$$

Residue

See: f'/f has a first order pole at a with residue N = order of zero.

Let $a_1, \dots, a_n$ denote the zeroes of f inside $C_r(z_0)$. Say mult. of a_k is m_k. Now

$$\frac{f'}{f} - \sum_{k=1}^{n} \frac{m_k}{(z-a_k)} = h \leftarrow \text{removable sing at each } a_k.$$

Give h values to make it analytic on $D_\rho(z_0)$ where $r<\rho<R$.

Cauchy's Thm: $$0 = \int_{C_r(z_0)} h\,dz = \int_{C_r(z_0)} \frac{f'}{f}\,dz - \sum_{k=1}^{N} m_k \underbrace{\int_{C_r(z_0)} \frac{1}{z-a_k}\,dz}_{2\pi i}$$

Argument Princ #2:

$$\frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f'}{f}\,dz = \begin{pmatrix}\#\text{ zeroes } f\\ \text{inside } C_r\\ \text{with mult.}\end{pmatrix} - \begin{pmatrix}\#\text{ poles } f\\ \text{inside } C_r\\ \text{with order}\end{pmatrix}$$

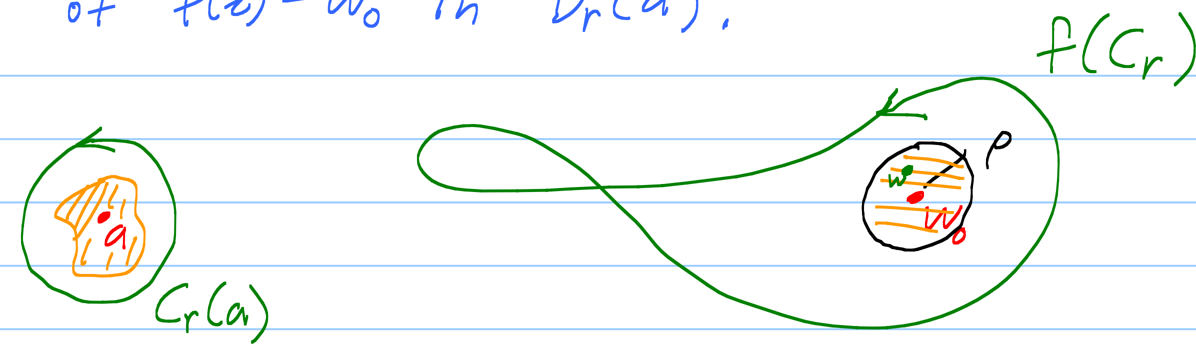
Hyp: f non-vanishing on $C_r(z_0)$, analytic on $D_R(z_0)$ except at finitely many poles inside $C_r(z_0)$.

Why: At a pole: $f(z) = (z-a)^{-N} F(z)$

$$\frac{f'(z)}{f(z)} = \frac{-N}{(z-a)} + \frac{F'(z)}{F(z)}$$

Pf of the OMT: f analytic on a domain Ω, non-const. Given $w_0 \in f(\Omega)$, there is an $a \in \Omega$ with $f(a) = w_0$. f not const $\Rightarrow$ $f(z) - w_0$ has an isolated zero at a.

So $\exists\, r > 0$ such that $\overline{D_r(a)} \subset \Omega$ and a is the <u>only</u> zero of $f(z) - w_0$ in $\overline{D_r(a)}$.



$w_0 = f(a)$ is not in $f(C_r(a)) \leftarrow$ compact set

$|f(z(t)) - w_0|$ cont fcn on $[a,b]$ non-vanishing.
It has a positive min $= m$. $\leftarrow$ dist $(w_0, f(C_r(a)))$

Pick ρ with $0 < \rho < m$. Note that $\overline{D_\rho(w_0)} \cap f(C_r(a)) = \emptyset$.

<u>Claim</u>: Every $w \in D_\rho(w_0)$ gets hit.

Let $F(z) = f(z) - w$. $\leftarrow$ no zeroes on $C_r(a)$

$$F'(z) = f'(z)$$

Arg Princ: $H(w) = \frac{1}{2\pi i} \int_{C_r(a)} \frac{f'(z)}{f(z) - w}\, dz = \left(\begin{array}{l} \#\text{ times} \\ f(z) = w \\ \text{inside } C_r(a) \end{array} \right)$

Aha! $H(w) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f'(z(t))\, z'(t)}{\underbrace{f(z(t))}\, - w}\, dt$ $\qquad f'(z(t))\, z'(t) = \frac{d}{dt} f(z(t))$

$$= \frac{1}{2\pi i} \int_{f(C_r(a))} \frac{1}{\zeta - w}\, d\zeta$$

$\boxed{\begin{array}{l} f(C_r(a)): \\ \zeta(t) = f(z(t)) \\ 0 \le t \le 2\pi \end{array}}$

HWK 2: Prob 1 $\Rightarrow$ $H(w)$ analytic!

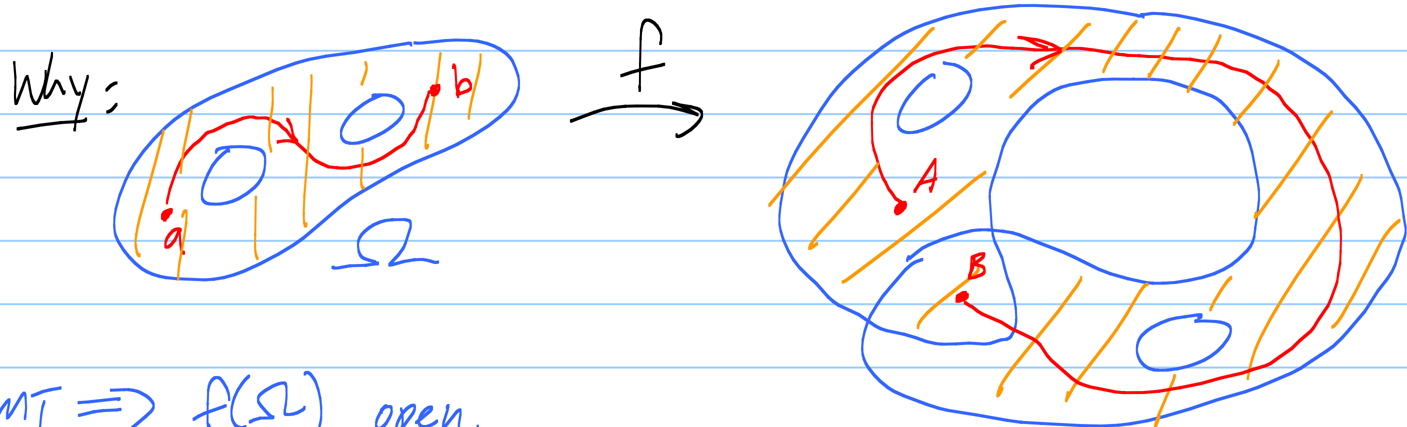
Integer valued analytic fcns on a disc must be const.!

$H(w_0) = 1$ or more. Conclude: Every w in $D_\rho(w_0)$ gets hit, and the same # times!

So $D_\rho(w_0) \subset f(\Omega)$ and we see $f(\Omega)$ is open.

Cor: Non-const analytic fcns map domains to domains.

Why:

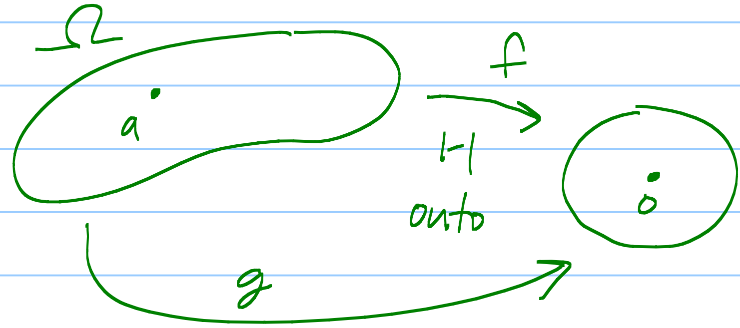


OMT $\Rightarrow f(\Omega)$ open.

Easy to see: Path conn. $\Omega \Rightarrow$ Path conn $f(\Omega)$.

Next time: OMT $\Rightarrow$ Inverse fcn thm for analytic fcns.

HWK prob 4:



$f(a) = 0$

$g(a) = 0$

Schwarz: $g(\underbrace{f^{-1}}(z)) = h \qquad h(0) = 0.$

need fact: f^{-1} is analytic

$|f'(a)| \geq |g'(a)|.$