Lecture 17 Inverse Function Theorems (I.F.T.)

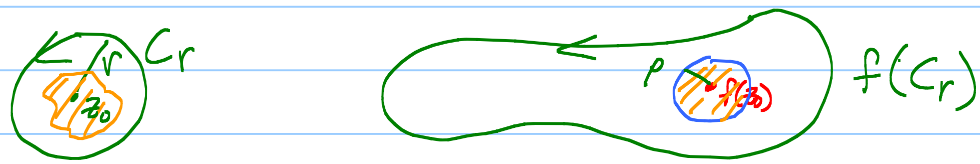
I.F.T.: Suppose f is analytic $D_r(z_0)$ and $f'(z_0) \neq 0$.

Then $\exists \rho > 0$ and a subdomain U of $D_r(z_0)$ $(z_0 \in U)$, such that $f : U \to D_\rho(f(z_0))$ is one-to-one and onto and f^{-1} is analytic on $D_\rho(f(z_0))$. Furthermore,

$$(f^{-1})'(w) = \frac{1}{f'(f^{-1}(w))} \quad \leftarrow \text{Chain rule}$$

$$(f^{-1})(f(z)) = z$$

Pf:



Go back to proof of OMT. Can shrink r so that z_0 is the only zero of $f(z) - f(z_0)$ in $\overline{D_r(z_0)}$.
[f not const because $f'(z_0) \neq 0$.] Can shrink r more to assure that f' is non-vanishing on $\overline{D_r(z_0)}$.

The order of the zero of $f(z) - f(z_0)$ at z_0 is one because $\frac{d}{dz}\left[f(z) - f(z_0)\right]\Big|_{z_0} = f'(z_0) \neq 0$.

OMT proof $\Rightarrow$ all $w \in D_\rho(f(z_0))$ get hit exactly once. f^{-1} exists.

Let $U = f^{-1}(D_\rho(f(z_0)))$. OMT $\Rightarrow$ f^{-1} is continuous.

$$DQ = \frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0}$$

$w = f(z)$ $\quad$ $z = f^{-1}(w)$

$w_0 = f(z_0)$

$$= \frac{z - z_0}{f(z) - f(z_0)} \to \frac{1}{f'(z_0)}$$ as $w \to w_0$ because $z = f^{-1}(w)$ is cont.

Get formula for $(f^{-1})'$ at w_0.

Super Inverse Function Theorem: Suppose f is analytic and one-to-one on a domain Ω. Then f' must be non-vanishing on Ω. IFT $\Rightarrow$ $f: \Omega \to f(\Omega)$ ← domain

$f^{-1}: f(\Omega) \to \Omega$ analytic.

Pf: Case $f'(z) \equiv C$. Then $f(z) = Cz + B$. Easy.

Otherwise zeroes of f' are isolated. Shrink $D_r(z_0)$ as above where $f'(z_0) = 0$. OMT gives a continuous inverse to f on $D_\rho(f(z_0))$. Aha! f^{-1} has non-vanishing derivative on $D_\rho(f(z_0)) - \{f(z_0)\}$ and is analytic there by the I.F.T.. R.R.S.T. $\Rightarrow$ $f(z_0)$ is removable! Write $F = f^{-1}$.

Chain rule:
$$F(f(z)) = z$$
$$F'(f(z))\underbrace{f'(z)}_{} = 1$$

↖ can't be zero at z_0.

No such z_0! f' is non-vanishing.

[Way false $\mathbb{R} \to \mathbb{R}$: $h(t) = t^3$ $\quad (-1,1) \xrightarrow[\text{onto}]{1\text{-}1} (-1,1)$

Inverse $\sqrt[3]{t}$ not diff'ble at 0. $(h'(0) = 0)$]

Remark: $f(x+iy) = u + iv$ $\quad \mathbb{R}^2 \to \mathbb{R}^2$

Jacobian Matrix
$$J = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} A & B \\ -B & A \end{bmatrix}$$

$u_y = -v_x$, $v_y = u_x$ $\quad$ C-R equs

$$\det J = u_x^2 + v_x^2 = |f'|^2$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Aha! $J^{-1} = \frac{1}{\det}\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$ ← see C-R eqns for f^{-1}

<u>Cor</u>: $\mathrm{Aut}(\Omega) = \{ f \text{ analytic on } \Omega ; f : \Omega \to \Omega \text{ 1-1 and onto} \}$

Automorphism <u>group</u> of Ω.

S.I.F.T. $\Rightarrow$ $\mathrm{Aut}(\Omega)$ <u>is</u> a group. [Need $f^{-1} \in \mathrm{Aut}(\Omega)$.]

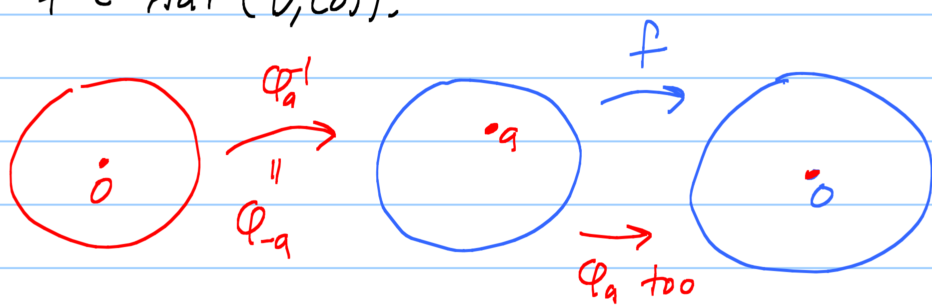
HWK 1: $\varphi_a(z) = \dfrac{z-a}{1-\bar{a}z}$ $a \in D_1(0)$. Is a Möbius Transformation.

<u>Facts</u>: 1) $\varphi_a : D_1(0) \to D_1(0)$ 1-1, onto, analytic.

2) $\varphi_a^{-1} = \varphi_{-a}$ ← $w = \dfrac{z-a}{1-\bar{a}z}$ Solve for z.

<u>Thm</u>: $\mathrm{Aut}(D_1(0)) = \{ \lambda \varphi_a : |\lambda| = 1 \text{ const.}, a \in D_1(0) \}$

<u>Pf</u>: Suppose $f \in \mathrm{Aut}(D_1(0))$. $a = f^{-1}(0)$



Now $f(\varphi_{-a}(z)) = F(z)$ is an auto with $F(0) = 0$.

Schwarz $\Rightarrow$ $|F'(0)| \le 1$. Aha! F^{-1} has all same prop. $|(F^{-1})'(0)| \le 1$

So $\left| \dfrac{1}{F'(0)} \right| \le 1$. So $|F'(0)| \ge 1$ too.

$|F'(0)|$ must $= 1$. Schwarz part 2 $\Rightarrow$

$$F(z) = \lambda z, \quad |\lambda| = 1.$$

$$f(\underbrace{\varphi_{-a}(z)}_{w}) = \lambda \underset{\varphi_a(w)}{\underset{\uparrow}{z}}$$

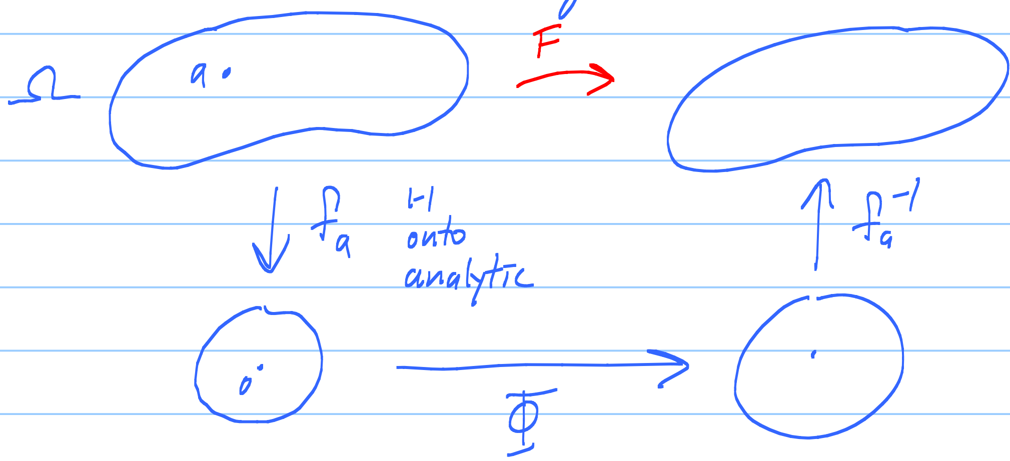
$w = \varphi_{-a}(z)$

$z = (\varphi_{-a})^{-1}(w) = \varphi_{-(-a)}(w) = \varphi_a(w)$

$$f(w) = \lambda \varphi_a(w) \quad \checkmark$$

<u>Remark</u>: Riemann Mapping Thm $\quad \Omega \neq \mathbb{C}$, simply connection domain

(no holes)



$$\mathrm{Aut}(\Omega) \cong \mathrm{Aut}(D_1(0))$$

<u>Local Mapping Theorem</u>: Suppose f analytic, <u>not</u> constant. $f(z_0) = w_0$. Then $f(z) - f(z_0)$ has an isolated zero at z_0, say of order N. [f takes value w_0 with mult N at z_0.]

$$f(z) - w_0 = (z - z_0)^N H(z) \quad \text{where } H(z_0) \neq 0.$$

Hmmm. Shrink r so H non-vanishing on $D_r(z_0)$.

Let $h(z)$ be an analytic N-th root of H there.

$$f(z) - w_0 = (z - z_0)^N [h(z)]^N = [\underbrace{(z - z_0) h(z)}_{g(z)}]^N$$

$$g'(z) = 1 \cdot h(z) + (z - z_0) h'(z)$$

$$g'(z_0) = h(z_0) \neq 0 \, !$$

$$f(z) = w_0 + g(z)^N$$