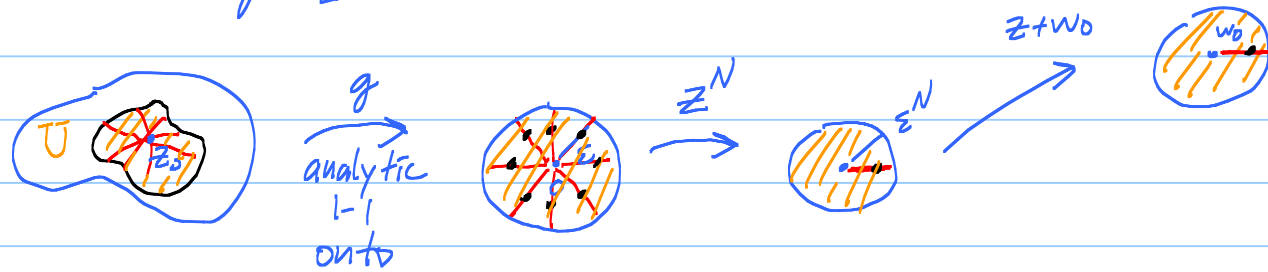
Lecture 18 Local Mapping Thm, Conformality, Residue Thm

$f(z_0) = w_0 \qquad f(z) - w_0 = (z-z_0)^N \underbrace{H(z)}_{h(z)^N} \qquad H(z_0) \neq 0$

$$= [\underbrace{(z-z_0)h(z)}_{g(z)}]^N \qquad g'(z_0) = h(z_0) \neq 0$$

$f(z) = w_0 + [g(z)]^N$



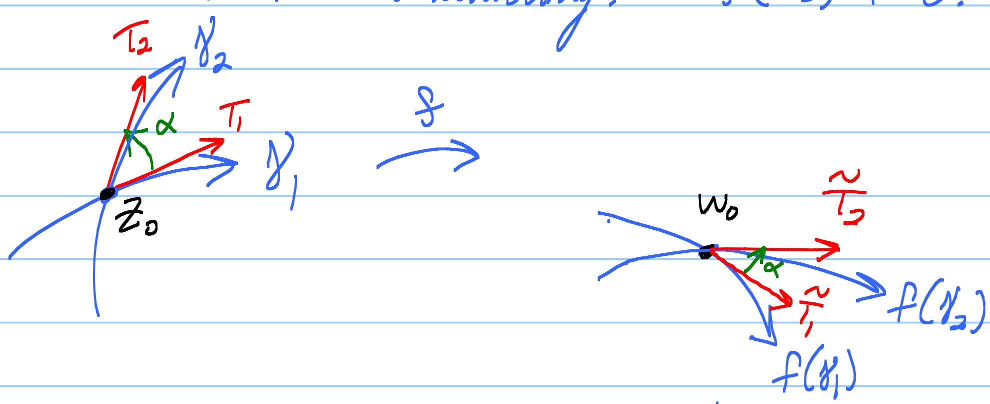
See $f^{-1}(w_0) = \{z_0\}$, but $f^{-1}(w) = \{N \text{ distinct points in } U\}$ if $w \neq w_0$.

$f: U - \{z_0\} \longrightarrow D_{\varepsilon^N}(w_0) - \{w_0\}$ is "N-sheeted covering map"

Alternate Pf of Super inverse fcn Thm: f 1-1 implies $N=1$ in local mapping thm. So $f(z) = w_0 + g(z)$. $g'(z_0) \neq 0$!

Big Fact: One-to-one analytic fcns are conformal.

Why: f 1-1 $\Rightarrow$ f' non-vanishing. $f'(z_0) \neq 0$.



$\gamma_j : z_j(t)$

$z_j(0) = z_0$

$f(\gamma_j): f(z_j(t))$

$T_j = z_j'(0) \qquad \tilde{T}_j = \frac{d}{dt} f(z_j(t))\Big|_{t=0} = f'(z_j(t))\, z_j'(t)\Big|_{t=0} = f'(w_0) T_j$

Aha! $f'(z_0) = re^{i\theta}$: $\tilde{T}_j = \underbrace{re^{i\theta}}\, T_j$

stretch by r
rotated c.c. by θ.

Angle α and sense is preserved.

Converse True! $f(x+iy) = u(x,y) + iv(x,y)$, u, v C^1-smooth,

$\det \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \neq 0$. If f is conformal, it is analytic!

Pf: $J = \underbrace{\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}}_{\text{causes rotation by } \theta} = \underbrace{\begin{bmatrix} R\cos\theta & -R\sin\theta \\ R\sin\theta & R\cos\theta \end{bmatrix}}_{\text{H.S. analytic geom.}}$ See C-R eqns!

Baby Residue Theorem Thm on Toy Regions.

Def^n: Suppose f has a pole at a with princ. part

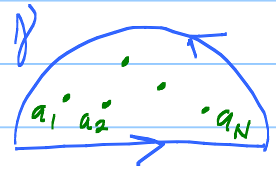
$$R(z) = \frac{A_N^{\neq 0}}{(z-a)^N} + \cdots + \frac{A_1}{z-a}$$

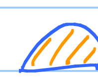
A_1 ← $\text{Res}_a f$ "Residue of f at a."

Key: 1) $\int_\gamma h\, dz = 0$, γ closed, h analytic and we have Cauchy's Thm.

2) $n \neq 1$: $\displaystyle\int_\gamma \frac{1}{(z-a)^n}\, dz = \int_\gamma \frac{d}{dz}\left[\frac{\left(\frac{-1}{n-1}\right)}{(z-a)^{n-1}}\right] dz = 0$

if γ closed.

EX: 

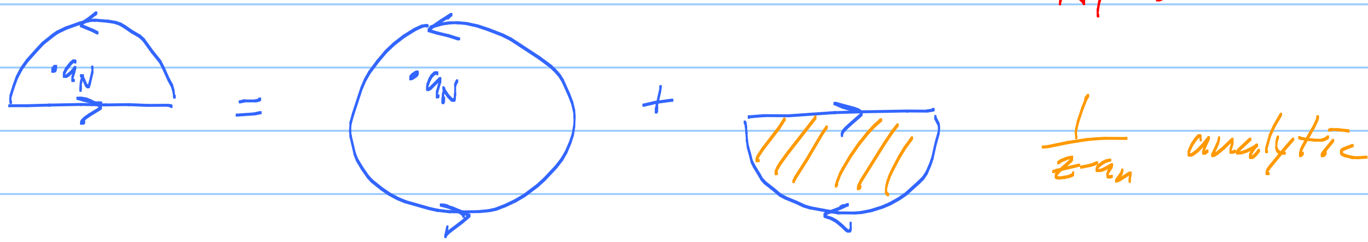
f analytic on open set containing  except at finitely many pts a_n inside γ where it has poles.

Then $\int_\gamma f\,dz = 2\pi i \sum_{n=1}^{N} \operatorname{Res}_{a_n} f$

Pf: Let $h = f - \sum_{n=1}^{N} R_n$ where $R_n =$ princ part of f at a_n

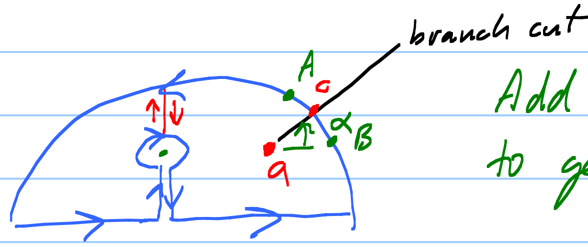
has removable sing at each a_n. Can use Cauchy's on convex to see that $0 = \int_\gamma h\,dz = \int_\gamma f\,dz - \sum_{n=1}^{N} \int_\gamma R_n\,dz$

But $\int_\gamma R_n\,dz = \underbrace{\int_\gamma \frac{A_N}{(z-a_n)^N}\,dz}_{=0} + \underbrace{\cdots}_{\cdots} + \underbrace{\int_\gamma \frac{A_1}{z-a_n}\,dz}_{A_1 \cdot 2\pi i}$



$= \quad 2\pi i \quad + \quad 0 \leftarrow$ by Cauchy on convex.

Toy Regions:



Add up Cauchy on convex to get Cauchy on Toys.

$\log_\alpha(z-a) \qquad \frac{d}{dz}\log_\alpha(z-a) = \frac{1}{z-a}$

γ_A^B be γ starting at A and going to B.

$$\int_\gamma \frac{1}{z-a}\,dz = \lim_{\substack{A \to C \\ B \to C}} \int_{\gamma_A^B} \frac{1}{z-a}\,dz = \lim \int_{\gamma_A^B} \frac{d}{dz}\log_\alpha(z-a)\,dz$$

$$\lim\Big(\log_\alpha(B-a) - \log_\alpha(A-a)\Big) = \Big(\ln|C-a| + i(\alpha+2\pi)\Big) - \Big(\ln|C-a| + i\alpha\Big)$$

$$= 2\pi i$$

Now let $H(w) = \int_\gamma \frac{1}{z-w}\,dz$. Know $H(a) = 2\pi i$.

analytic

$$H'(w) = \int_\gamma \frac{1}{(z-w)^2}\,dz = \int_\gamma \frac{d}{dz}\left[\frac{-1}{z-w}\right] dz = 0$$

$H \equiv$ const., which must be $2\pi i$

Fact: Cauchy Theorem and Residue Thm hold on Toy regions.