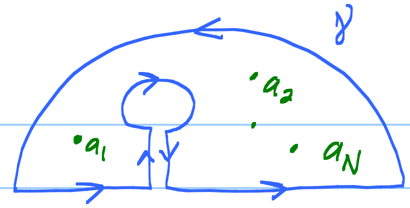
Lecture 19 Residue Theorem



$$\int_\gamma f\,dz = 2\pi i \sum_{n=1}^{N} \operatorname{Res}_{a_n} f$$

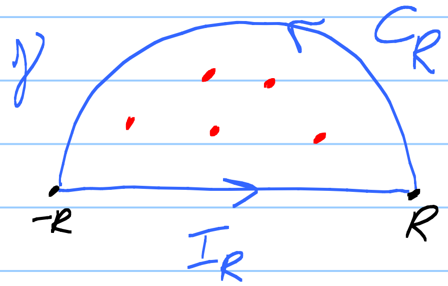
Theorem: Suppose P and Q are polys and

1) $\deg Q \geq \deg P + 2$
2) Q has no zeroes on $\mathbb{R}$

Then
$$\int_{-\infty}^{\infty} \frac{P(t)}{Q(t)}\,dt = 2\pi i \sum_{a\in Z_Q^+} \operatorname{Res}_a \frac{P}{Q}$$

where Z_Q^+ = the set of zeroes of Q in the UHP $= \{z : \operatorname{Im} z > 0\}$.

Pf:



All zeroes of Q in Z_Q^+ in the "D" for $R > R_0$.

$$\int_\gamma \frac{P}{Q}\,dz = 2\pi i \sum_{a\in Z_Q^+} \operatorname{Res}_a \frac{P}{Q}$$

I_R : $z(t) = t$; $-R \leq t \leq R$

$$\int_{I_R} \frac{P}{Q}\,dz = \int_{-R}^{R} \frac{P(t)}{Q(t)} \cdot 1 \cdot dt \longleftarrow \text{want}$$

Basic Poly Est ;
$$a|z|^{\deg P} \leq |P(z)| \leq A|z|^{\deg P}, \quad |z| > R_P$$
$$b|z|^{\deg Q} \leq |Q(z)| \leq B|z|^{\deg Q}, \quad |z| > R_Q$$

$$\left|\int_{C_R} \frac{P}{Q}\,dz\right| \leq \left(\operatorname{Max}_{C_R}\left|\frac{P}{Q}\right|\right) \cdot \pi R$$

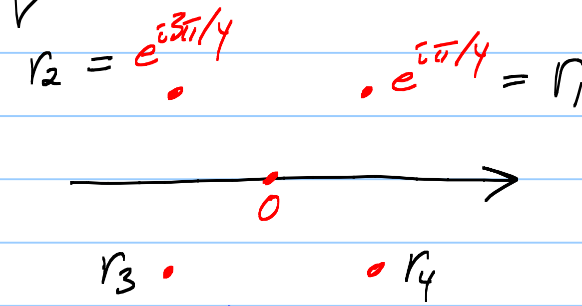
$$\leq \frac{A R^{\deg P}}{b R^{\deg Q}} \cdot \pi R = \frac{\pi A}{b} \frac{1}{R^{\underbrace{\deg Q - \deg P - 1}_{\geq 1}}}$$

$$\longrightarrow 0 \quad \text{as} \quad R \to \infty.$$

$$\int_{I_R} + \int_{C_R} = 2\pi i \sum \text{Res}$$

$$\downarrow \qquad \downarrow$$

$$\int_{-\infty}^{\infty} \frac{P}{Q}\, dt + 0 =$$



EX: $\displaystyle\int_{-\infty}^{\infty} \frac{1}{t^4+1}\, dt$

$$= 2\pi i \left[\text{Res}_{e^{i\pi/4}} \frac{1}{z^4+1} + \text{Res}_{e^{i3\pi/4}} \frac{1}{z^4+1} \right]$$

$$\frac{1}{z^4+1} = \frac{1}{\prod_{j=1}^{4} (z-r_j)} = \frac{1}{(z-r_1)} \Bigg[\underbrace{\frac{1}{\prod_{j\neq 1} (z-r_j)}}_{\text{analytic near } r_1} \Bigg]$$

$$H(z) = A_0 + A_1(z-r_1) + \cdots$$

$$= \underset{\text{"Simple pole"}}{\frac{A_0}{z-r_1}} + \underbrace{A_1 + A_2(z-r_1) + \cdots}_{\text{analytic}}$$

$$A_0 = \text{Res}_{r_1} \frac{1}{z^4+1} = \text{value of } H \text{ at } r_1 = \frac{1}{\prod_{j\neq 1} (r_1 - r_j)}$$

etc. ...

Fact: Suppose f and g analytic near a and $f(z)$ has a simple zero at a (meaning of order 1).

Then $\operatorname{Res}_a \frac{g}{f} = \frac{g(a)}{f'(a)}$. $\left[\underbrace{f(a)=0,\ f'(a)\neq 0}_{\text{a simple zero}}\right]$

Why: $f(z) = a_0 + a_1(z-a) + \cdots = (z-a)\underbrace{\left[a_1 + a_2(z-a) + \cdots\right]}_{H(z) \text{ analytic}}$

$a_0 = f(a) = 0$ $\quad a_1 = \frac{f'(a)}{1!}$

$H(a) = a_1 = f'(a)$,

$$\frac{g}{f} = \frac{g}{(z-a)H} = \frac{1}{z-a}\Bigg[\underbrace{\frac{g(z)}{H(z)}}_{A_0 + A_1(z-a)+\cdots}\Bigg]$$

analytic near a

$$= \frac{A_0}{z-a} + \underbrace{A_1 + A_2(z-a) + \cdots}_{\text{analytic near } a}$$

See $\operatorname{Res}_a \frac{g}{f} = A_0 = \frac{g(a)}{H(a)} = \frac{g(a)}{f'(a)}$ ✓

Back to $\operatorname{Res}_{e^{i\pi/4}} \frac{1}{z^4+1}$ $\quad \leftarrow g(z) \equiv 1$, $\leftarrow f(z) = z^4+1$

$f(e^{i\pi/4}) = 0$

$f'(z) = 4z^3$ $\quad f'(e^{i\pi/4}) = 4e^{i3\pi/4}$

not zero!

$$= \frac{1}{4e^{i3\pi/4}} = \frac{1}{4}e^{-i3\pi/4}$$

$$\int_{-\infty}^{\infty} = 2\pi i\left[\frac{1}{4e^{i3\pi/4}} + \frac{1}{4e^{(i3\pi/4)\cdot 3}}\right]$$

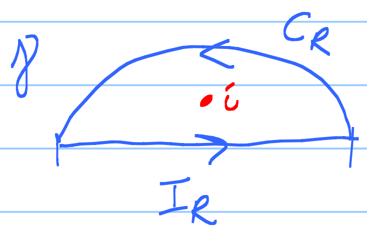
$$= \frac{\pi i}{2}\left[e^{-i3\pi/4} + e^{-i9\pi/4}\right] = \frac{\pi i}{2}\left(\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\right) = \frac{\pi}{\sqrt{2}}$$

Fourier Transform: $\hat{f}(s) = \int_{-\infty}^{\infty} e^{isx} f(x)\,dx$

EX: $$I(s) = \int_{-\infty}^{\infty} \frac{e^{isx}}{x^2+1}\,dx, \quad \underline{s > 0}.$$

$$g(z) = \frac{e^{isz}}{z^2+1} \qquad \left|e^{is(x+iy)}\right| = \left|e^{isx}e^{-sy}\right| = e^{-sy} < 1$$

if $y > 0$



$$\int_\gamma g\,dz = 2\pi i \operatorname{Res}_i \frac{e^{isz}}{z^2+1}$$

$$= 2\pi i \frac{e^{is\cdot i}}{2(i)} = \pi e^{-s}$$

$$\underbrace{\int_{I_R} g\,dz}_{\text{want}} + \underbrace{\int_{C_R} g\,dz}_{\to 0 \text{ as } R\to\infty}$$

$$\left|\int_{C_R} g\,dz\right| \le \left(\max_{C_R} \frac{|e^{isz}|}{|z^2+1|}\right)\cdot \pi R \le \frac{1}{R^2-1}\cdot \pi R$$

$\uparrow$ if $R > 1$

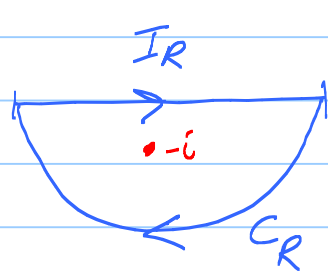
$$\le \frac{\pi R}{R^2-1} \to 0 \text{ as } R\to\infty.$$

<u>Note</u>: $$|z^2+1| = |z^2-(-1)| \ge \big|\,|z^2| - |-1|\,\big|$$

$$|R^2-1| = R^2-1 \text{ if } R > 1.$$

<u>Case $s<0$</u>:

$|e^{isz}| \le 1$ in LHP

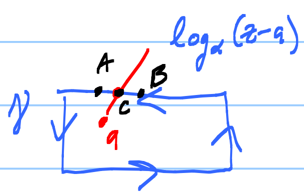


Clockwise!

$$\int_\gamma g\,dz = -2\pi i \operatorname{Res}_{-i}\, g$$

$\uparrow$!

<u>Bell</u>:

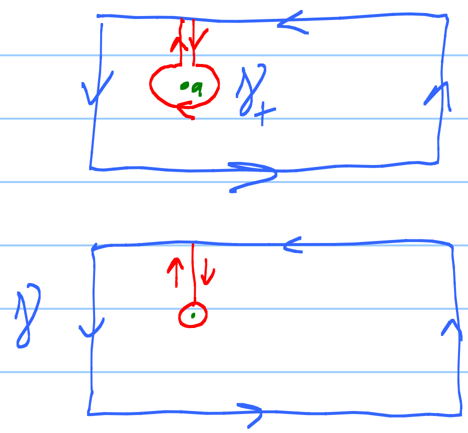


branch cut that crosses γ at exactly one point transversely.

$$\int_\gamma \frac{1}{z-a}\,dz = 2\pi i.$$

Let a become w.
Get $H(w)$ with $H'(w) \equiv 0$.

Stein: Keyhole argument



Cauchy on Toys yields

$$\int_{\gamma_+} \frac{1}{z-a}\,dz = 0$$

$$\int_\gamma \frac{1}{z-a}\,dz = -\int_{\odot} \frac{1}{z-a}\,dz$$

$$z(t) = a + \varepsilon e^{it}$$

Cauchy Int Formula: Replace 1 in numerator by $f(z)$.