Lecture 2

Def^n: f on $D_r(z_0)$ is *complex differentiable* at z_0 provided that

$$\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} \text{ exists.} \quad \text{Call limit } f'(z_0).$$

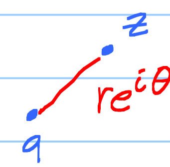
EX: $f(z)=z^2$. $\quad DQ = \dfrac{z^2-a^2}{z-a} \to 2a$ as $z \to a$. $\quad f'(a)=2a \quad [f'(z)=2z]$

Def^n: $z = x+iy$. Conjugate of z: $\bar{z} = x-iy$

Facts: $z \cdot \bar{z} = x^2+y^2 = |z|^2$. $\quad \overline{zw} = \bar{z}\cdot\bar{w}$, $\quad \overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad \overline{\left(e^{i\theta}\right)} = \cos\theta - i\sin\theta = e^{-i\theta}$$

EX: $\bar{z} = x-iy$ is not complex diff. $\qquad z-a = re^{i\theta}$



$$DQ = \frac{\bar{z}-\bar{a}}{z-a} = \frac{\left(re^{-i\theta}\right)}{re^{i\theta}} = e^{-2i\theta} \longleftarrow \text{no limit as } z \to a.$$

Def^n: f is *analytic* (or holomorphic) on an *open* set if it is $\mathbb{C}$-diff'ble at each point in the set.

Def^n: f is continuous at z_0 means $\lim_{z\to z_0} f(z) = f(z_0)$.

Thm: Analytic fcns are continuous.

Pf: $\dfrac{f(z)-f(a)}{z-a} = f'(a) + E(z)$ $\qquad$ $f'(a)$ exists means $\lim_{z\to a} E(z) = 0$.

$$f(z) = \underbrace{f(a) + f'(a)(z-a)}_{\mathbb{C}\text{- linear}} + \underbrace{E(z)(z-a)}_{\text{very small as } z\to a} \qquad \text{Define } E(a)=0.$$

$z \to a$: $\quad f(a) + f'(a)\cdot 0 + 0\cdot 0 = f(a)$

Cauchy-Riemann Eqns: (CR-Eqns) Assume $f(x+iy) = u(x,y) + i\,v(x,y)$ is $\mathbb{C}$-diff'ble at $z_0 = x_0+iy_0$. Then first order partial derivs of u and v wrt x and y exist at (x_0, y_0) and

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases} \quad \text{(C-R Eqns) hold at } (x_0, y_0).$$

Notation: $u_x = \dfrac{\partial u}{\partial x}$, etc. $\quad u^0 = u(x_0, y_0)$, etc.

$u_x^0 = u_x(x_0, y_0)$, etc.

Pf: Step 1: Slide z into z_0 along horiz:

$x_0 + iy_0 \quad x + iy_0$

$z_0 \leftarrow z$

$$DQ = \frac{f(z) - f(z_0)}{z - z_0} = \frac{(u + iv) - (u^0 + iv^0)}{(x + iy_0) - (x_0 + iy_0)} = \underbrace{\frac{u(x, y_0) - u(x_0, y_0)}{x - x_0}}_{DQ_1} + i \underbrace{\frac{v(x, y_0) - v(x_0, y_0)}{x - x_0}}_{DQ_2}$$

Baby Thm $\Rightarrow$ DQ_1 and DQ_2 have limits as $x \to x_0$, i.e., partials u_x and v_x exist, and

$$\boxed{f'(z_0) = u_x(x_0, y_0) + i\, v_x(x_0, y_0)}$$

Step 2:

$z = x_0 + iy$

$\downarrow$

$z_0 = x_0 + iy_0$

$$DQ = \frac{[u(x_0, y) + iv(x_0, y)] - [u^0 + iv^0]}{(x_0 + iy) - (x_0 + iy_0)}$$

$$= \frac{v(x_0, y) - v(x_0, y_0)}{y - y_0} - i\, \frac{u(x_0, y) - u(x_0, y_0)}{y - y_0}$$

So v_y and u_y exist at (x_0, y_0) and and

$$\boxed{f'(z_0) = v_y(x_0, y_0) - i\, u_y(x_0, y_0)}$$

Step 3: Equate red boxes. Get C-R Eqns.

EX: $E(x+iy) = \underbrace{e^x \cos y}_{u} + i\underbrace{e^x \sin y}_{v}$ C-R Eqns hold!

Is $E(z)$ analytic? Yes, because

Partial Converse to C-R Eqns: Suppose u and v are continuous, and first partials exist and are continuous. (C^1-smooth)
And suppose C-R Eqns hold. Then $f(x+iy) = u(x,y) + i\,v(x,y)$ is analytic. [Ω open set in background.]

Looman-Menchoff Thm: Only need to assume

1) u, v continuous
2) first partials exist $\implies f$ analytic
3) C-R Eqns hold.

Narasimhan: Complex Analysis in one var p. 43-50.

Calculus fact: Taylor's formula in $\mathbb{R}^2$: u is C^1-smooth.

$$u(x,y) = u(x_0,y_0) + u_x(x_0,y_0)(x-x_0) + u_y(x_0,y_0)(y-y_0) + R(x,y)$$

where $R(x,y) \to 0$ so fast as $(x,y) \to (x_0,y_0)$ that

$$\lim_{(x,y)\to(x_0,y_0)} \frac{R(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0 \text{ too.}$$

[Pf. See 530 Home Page]

Pf of partial converse. $DQ = \dfrac{f(z)-f(z_0)}{z-z_0} = \dfrac{(u+iv)-(u^0+iv^0)}{z-z_0}$

CR Eqns

$$= \frac{\left[u_x^0(x-x_0) + \overset{-v_x^0}{u_y^0}(y-y_0)\right] - i\left[v_x^0(x-x_0) + \overset{u_x^0}{v_y^0}(y-y_0)\right]}{z-z_0} + \frac{R_u + iR_v}{z-z_0}$$

$$= \frac{\left[u_x^0 + i v_x^0\right]\left[(x-x_0) + i(y-y_0)\right]}{z-z_0} + \frac{R}{z-z_0}$$

$$= [u_x^0 + i v_x^0] + \underbrace{\frac{R}{z - z_0}}_{\mathcal{E}}$$

where $|\mathcal{E}| = \dfrac{|R|}{|z-z_0|} \leq \dfrac{|R_u| + |R_v|}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} \to 0$ as $z \to z_0$.

So $DQ \to u_x^0 + i v_x^0$ and f is $\mathbb{C}$-diff'ble at (x_0, y_0).
So f is analytic.

Cor: $E(z)$ is analytic on $\mathbb{C}$. $\leftarrow$ $E(z)$ is an entire fcn.

Remark: Proofs of freshman calc facts about limits and derivatives all hold $\mathbb{C} \to \mathbb{C}$ with same proofs.

$$(fg)' = f'g + fg', \qquad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \text{ provided } g(z) \neq 0.$$

HWK: Complex chain rules holds: [Baby Rudin $\mathbb{R} \to \mathbb{R}$ goes $\mathbb{C} \to \mathbb{C}$ easily.]

Fact: Complex polys are entire.