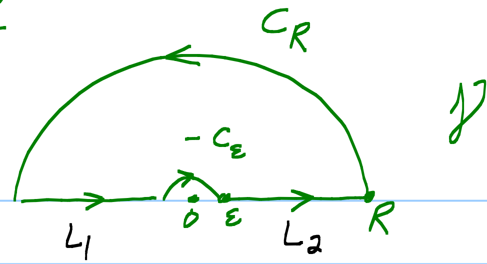
Lecture 20

$$I = \int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}$$



$$g(z) = \frac{\sin z}{z} \quad \text{Nope!}$$

$$f(z) = \frac{e^{iz}}{z} \quad \text{Yes!} \qquad f(t) = \frac{e^{it}}{t} = \underbrace{\frac{\cos t}{t}}_{\text{odd}} + i \underbrace{\frac{\sin t}{t}}_{\text{even}}$$

$$\left(\int_{L_1} + \int_{L_2}\right) f(t)\, dt = 0 + i\, 2\int_\varepsilon^R \frac{\sin t}{t}\, dt \leftarrow \text{want}$$

On $-C_\varepsilon$:

$$\frac{e^{iz}}{z} = \frac{1 + iz + \frac{(iz)^2}{2!} + \cdots}{z} = \frac{1}{z} + \underbrace{\left[i + \frac{i^2}{2!} z + \frac{i^3}{3!} z^2 + \cdots\right]}_{H(z) \text{ entire!}}$$

$$\int_{-C_\varepsilon} \frac{e^{iz}}{z}\, dz = \underbrace{\int_{-C_\varepsilon} \frac{1}{z}\, dz}_{} + \underbrace{\int_{-C_\varepsilon} H(z)\, dz}_{}$$

$$= -\int_{C_\varepsilon} \frac{1}{z}\, dz \qquad |S| \le \Big(\max_{C_\varepsilon} |H|\Big)(\pi\varepsilon) \to 0 \text{ as } \varepsilon \to 0$$

Take $\varepsilon < 1$. $\quad \max_{C_\varepsilon}|H| \le M$

$$= -\int_0^\pi \frac{1}{\varepsilon e^{it}}\, i\varepsilon e^{it}\, dt$$

$$= -\pi i$$

On C_R:

$$\left|\int_{C_R} \frac{e^{iz}}{z}\, dz\right| \le \left(\max_{C_R} \left|\frac{e^{iz}}{z}\right|\right)(\pi R)$$

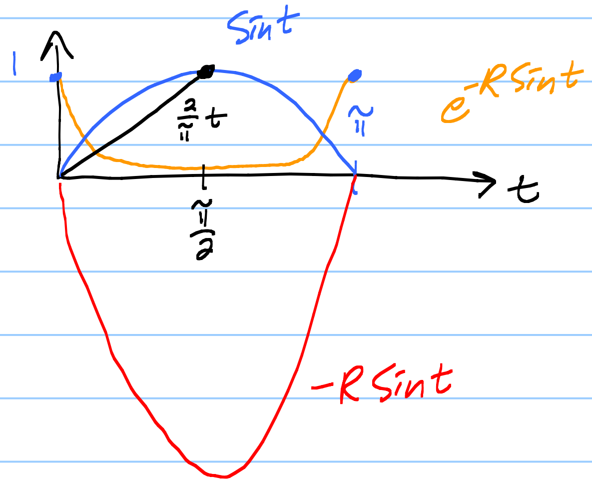
$$\le \underbrace{\frac{1}{R} \cdot \pi R}_{=\pi} \quad \text{Darn!}$$

$|e^{iz}| \le 1$ on UHP

Jordan's Lemma: $\quad C_R: z(t) = Re^{it} \quad 0 \le t \le \pi$

$$\left| \int_{C_R} \frac{e^{iz}}{z}\, dz \right| = \left| \int_0^{\pi} \frac{e^{i(R\cos t + iR\sin t)}}{Re^{it}} \underbrace{iRe^{it}\, dt}_{z'(t)\,dt} \right|$$

$$= \left| \int_0^{\pi} \underset{\text{modulus} = 1}{e^{iR\cos t}}\, e^{-R\sin t}\, dt \right| \le \underbrace{\int_0^{\pi} e^{-R\sin t}\, dt}$$



$$= 2\int_0^{\pi/2} e^{-R\sin t}\, dt$$

Aha! $0 \le \frac{2}{\pi} t \le \sin t$ on $[0, \frac{\pi}{2}]$

$$-R\sin t \le -R\tfrac{2}{\pi} t \le 0$$

$$e^{-R\sin t} \le e^{-\frac{2R}{\pi} t} \le 1$$

So $$\int_0^{\pi/2} e^{-R\sin t}\, dt \le \int_0^{\pi/2} e^{-\frac{2R}{\pi} t}\, dt = \left[-\frac{\pi}{2R} e^{-\frac{2R}{\pi} t} \right]_0^{\pi/2}$$

$$= \frac{\pi}{2R}\left(1 - e^{-R}\right) < \frac{\pi}{2R} \xrightarrow[\text{as } R \to \infty]{} 0$$

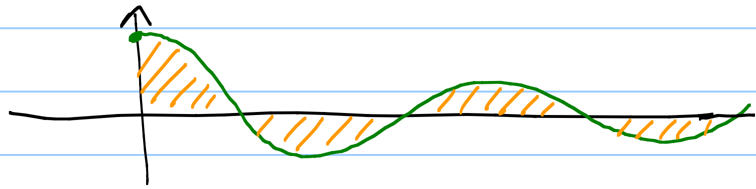
Last step: Cauchy Thm on Toys : $\int_{\gamma} \frac{e^{iz}}{z}\, dz = 0$

$$\Bigg(\underbrace{\int_{L_1} + \int_{L_2}}_{2i\int_{\varepsilon}^{R} \frac{\sin t}{t} dt} + \int_{-C_\varepsilon} + \int_{C_R} \Bigg) \frac{e^{iz}}{z}\, dz = 0$$

Let $\varepsilon \to 0$, $R \to \infty$.

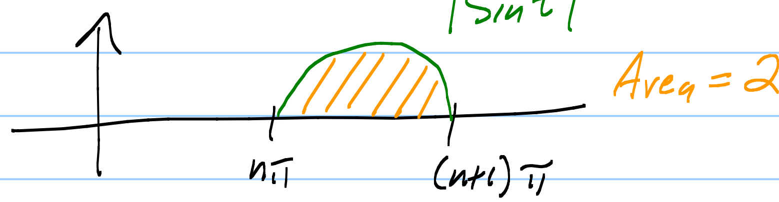
$$2i\int_0^{\infty} \frac{\sin t}{t}\, dt + -\pi i + 0 = 0$$

So $\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$ $\left(\text{and } \int_{-\infty}^{\infty} \frac{\sin t}{t}\,dt = \pi\right)$



Alternating series test shows $\lim_{R\to\infty} \int_0^R \frac{\sin t}{t}\,dt$ exists.

But $\int_0^\infty \left|\frac{\sin t}{t}\right| dt = \infty$



$\int_{n\pi}^{(n+1)\pi}$

$$\frac{|\sin t|}{\pi(n+1)} \le \left|\frac{\sin t}{t}\right| \le \frac{|\sin t|}{n\pi}$$

$$\frac{2}{\pi(n+1)} \le \int_{n\pi}^{(n+1)\pi} \left|\frac{\sin t}{t}\right| dt \le \frac{2}{\pi n}$$

So $\int_0^N \left|\frac{\sin t}{t}\right| dt \sim \sum_1^N \frac{1}{n} \sim \ln N$

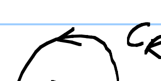
Theorem: P, Q polynomials, Q has no zeroes on $\mathbb{R}$, $\deg Q \ge \deg P + 1$.

Then $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{ix}\,dx = 2\pi i \sum_{a\in Z_Q^+} \mathrm{Res}_a\left(\frac{P(z)}{Q(z)} e^{iz}\right)$

where $Z_Q^+ = \{\text{zeroes of } Q \text{ in UHP}\}$.

Pf: Keys: 1) Basic poly estimate $\left|\frac{P(z)}{Q(z)}\right| \le \frac{C}{|z|}$ in $|z|>R_0$.

Ouch! Basic estimate fails us on S_{C_R}.



But Jordan's Lemma shows $\int_{C_R} \to 0$.

EX: $\int_0^{2\pi} \sin^4\theta \, d\theta$ $\quad \sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i} = \dfrac{e^{i\theta} - \frac{1}{e^{i\theta}}}{2i}$

Hmmm. $C_1 : z(\theta) = e^{i\theta} \quad 0 \le \theta \le 2\pi$

$z'(\theta) = ie^{i\theta}$

$$\int_0^{2\pi} \underbrace{\left[\frac{e^{i\theta} - \frac{1}{e^{i\theta}}}{2i}\right]}_{\frac{z-\frac{1}{z}}{2i}}^4 \frac{\overbrace{ie^{i\theta}\, d\theta}^{dz}}{\underbrace{ie^{i\theta}}_{iz}} = \int_{C_1} \left[\frac{z - \frac{1}{z}}{2i}\right]^4 \frac{1}{iz}\, dz$$

Residue term!

$$= \frac{1}{16}\int_{C_1} \left(z^4 - 4z^2 + 6 - 4\frac{1}{z^2} + \frac{1}{z^4}\right)\frac{1}{iz}\, dz$$

Method: $R(z,w)$ rational in z, w.

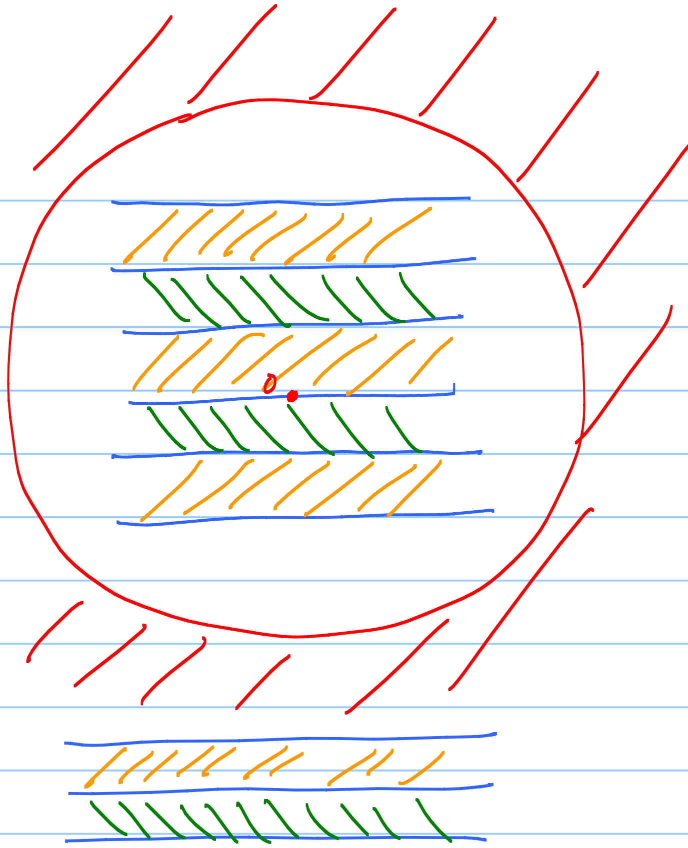
$$\int_0^{2\pi} R(\cos t, \sin t)\, dt = \int_{C_1} R\left(\frac{z+\frac{1}{z}}{2}, \frac{z-\frac{1}{z}}{2i}\right)\frac{1}{iz}\cdot dz$$

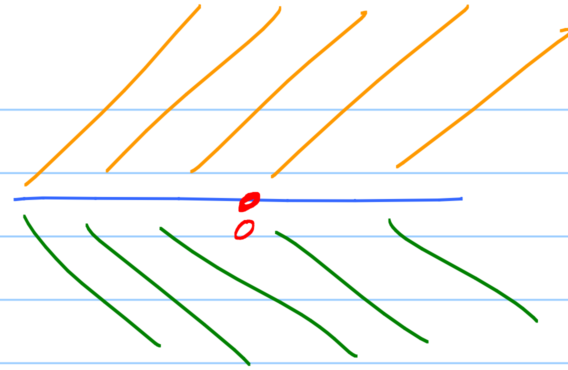
Use Residue Thm.

EX: $e^{1/z}$ has an essential sing at $z=0$.

$e^{f(z)}$ where $f(z)$ has a pole at $z=0$.

ess. sing at 0.





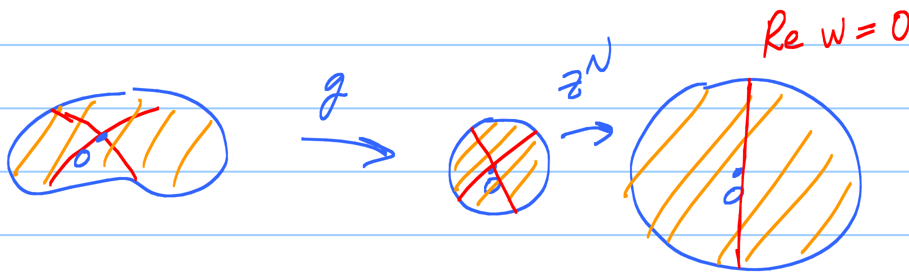
f(z) maps $D_\varepsilon(0) - \{0\}$

onto the outside of a big circle

Hints: 1) $\frac{1}{f(z)}$ has a zero at 0.

2) OMT (or local mapping thm).

Think about zeroes of harmonic fcns $u = \mathrm{Re}\, f$



Zeroes of harmonic fcns are curves that cross at angles $\frac{2\pi}{N}$!