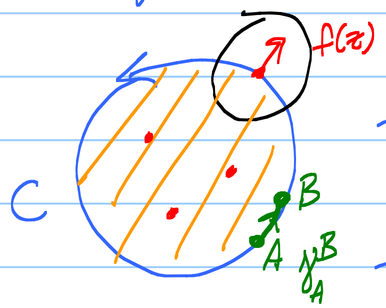
Lecture 21 Residue integrals

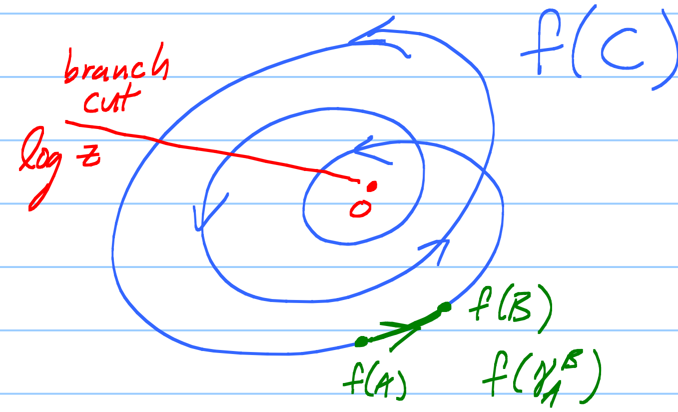
Why "Argument Principle"?

$$\frac{1}{2\pi i}\int_C \frac{f'}{f}\,dz = \left(\begin{array}{c}\#\text{ zeroes}\\ \text{of } f\\ \text{inside } C\end{array}\right)$$



f non-vanishing on C

$\xrightarrow{\ f\ }$



Fact: $\frac{d}{dz}(\log z) = \frac{1}{z}$

Why: $e^{\log z} = z$

Chain rule: $\underbrace{e^{\log z}}_{z}\,\frac{d}{dz}(\log z) = 1$

$$\frac{d}{dz}\left[\log f(z)\right] = \frac{f'(z)}{f(z)}$$

Careful: $\log(e^z) = z$ plus possible multiple of $2\pi i$

$$\int_{\gamma_A^B}\frac{f'}{f}\,dz = \int_{\gamma_A^B}\frac{d}{dz}\left[\log f\right]dz = \log f(z)\Big|_A^B$$

$$= \left[\operatorname{Ln}|f(B)| + i\arg f(B)\right] - \left[\operatorname{Ln}|f(A)| + i\arg f(A)\right]$$

$$= \left(\operatorname{Ln}|f(B)| - \operatorname{Ln}|f(A)|\right) + i\,\Delta_{\gamma_A^B}\arg f$$

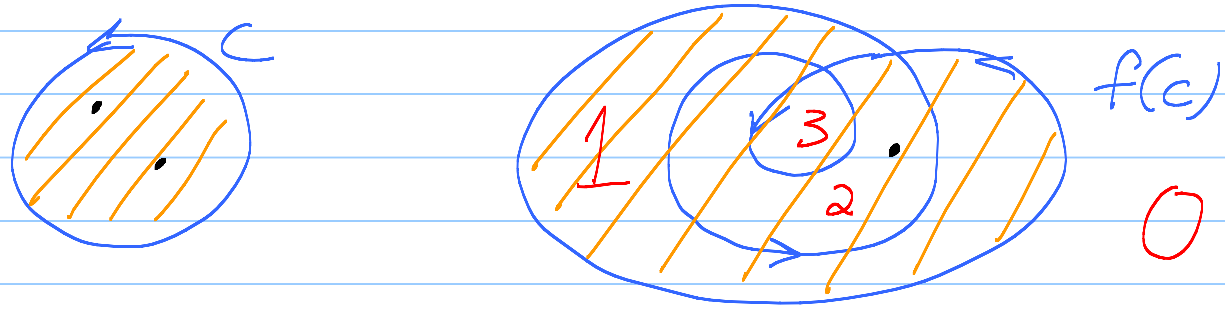
Add up segments around C. Real parts pairwise cancel.

$$\int_C \frac{f'}{f}\,dz = i\,\Delta_C \arg f$$

$$\frac{1}{2\pi i}\int_C \frac{f'}{f}\,dz = \frac{1}{2\pi}\,\Delta_C \arg f \longleftarrow$$ counts the number of times f spins around 0 counterclockwise.

Note: $$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)-w}\,dz = \begin{pmatrix}\#\text{ times}\\ f(z)=w\\ \text{inside } C\end{pmatrix}$$

$$= \frac{1}{2\pi i}\int_{f(C)} \frac{1}{\zeta-w}\,dw$$ ← winding number of $f(C)$ about w $= \operatorname{ind}_{f(C)} w$



Important fact:

$$H(w) = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)-w}\,dz$$ analytic (integer valued!)

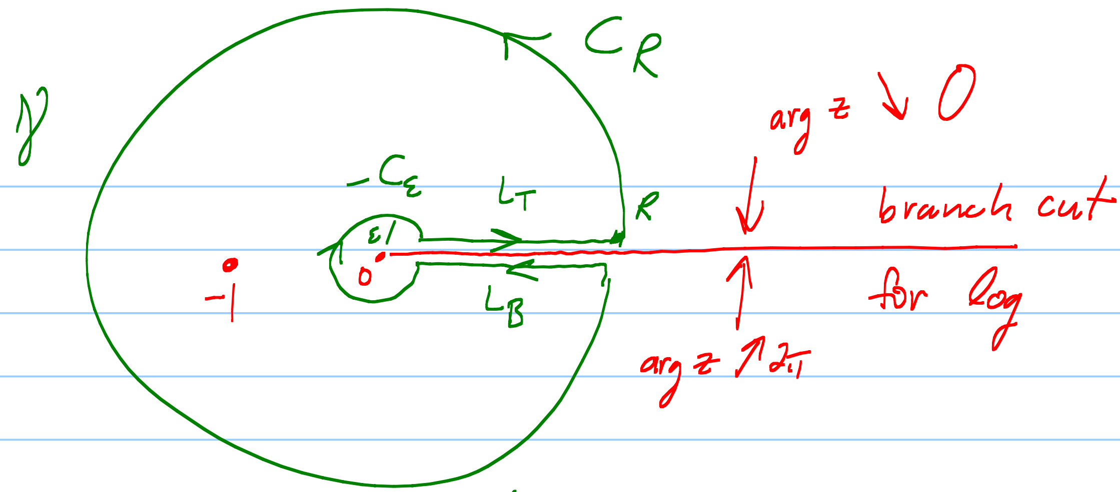
$$= \frac{1}{2\pi i}\int_{f(C)} \frac{1}{\zeta-w}\,d\zeta$$

$$H'(w) = \frac{1}{2\pi i}\int_{f(C)} \frac{1}{(\zeta-w)^2}\,d\zeta = \frac{1}{2\pi i}\int_{f(C)} \frac{d}{d\zeta}\left[\frac{-1}{(\zeta-w)}\right]d\zeta = 0$$

So $H(w)$ is constant on connected components of $\mathbb{C}-f(C)$

EX: $$I = \int_0^\infty \frac{x^{-\alpha}}{x+1}\,dx \qquad 0<\alpha<1$$

$$x^{-\alpha} = e^{-\alpha \ln x} \qquad z^{-\alpha} = e^{-\alpha \log z}$$ ← branch?



$$\int_\gamma \frac{e^{-\alpha \log z}}{z+1}\,dz = 2\pi i \operatorname{Res}_{-1} \frac{e^{-\alpha \log z}}{z+1} = 2\pi i\, e^{-\alpha \log(-1)}$$

$$= 2\pi i \left(e^{-\alpha(\ln|-1| + i\pi)}\right)$$

$$= 2\pi i\, e^{-i\alpha\pi}$$

Step 1: Let open mouth angle $\downarrow 0$.

$$\underbrace{\underbrace{\int_\varepsilon^R \frac{e^{-\alpha \ln t}}{t+1}\,dt}_{\int_{L_T} f(z)\,dz} + \underbrace{\int_\varepsilon^R \frac{e^{-\alpha(\ln t + i2\pi)}}{t+1}\,dt}_{\int_{L_B} f(z)\,dz}} + \int_{-C_\varepsilon} + \int_{C_R}$$

$$= \left(1 - e^{-\alpha 2\pi i}\right) \underbrace{\int_\varepsilon^R \frac{t^{-\alpha}}{t+1}\,dt}_{\text{want!}}$$

Next: $\left|e^{-\alpha \log z}\right| = \left|e^{-\alpha(\ln|z| + i \arg z)}\right| = e^{-\alpha \ln|z|} = |z|^{-\alpha}$

(Fact: $|z^{-\alpha}| = |z|^{-\alpha}$)

So $$\left|\int_{C_R} \frac{e^{-\alpha \log z}}{z+1}\,dz\right| \le \left(\frac{R^{-\alpha}}{R-1}\right)\cdot 2\pi R \to 0$$ as $R\to\infty$ because $\alpha>0$,

C_R: $|z+1| = |z-(-1)| \ge \big||z|-1\big| = |R-1| = R-1$ if $R>1$

$-C_\varepsilon$: $|z+1| \ge \big||z|-1\big| = |\varepsilon-1| = 1-\varepsilon$ if $\varepsilon<1$.

$$\left|\int_{-C_\varepsilon} f(z)\,dz\right| \le \frac{\varepsilon^{-\alpha}}{1-\varepsilon}\cdot 2\pi\varepsilon = \frac{2\pi\,\varepsilon^{1-\alpha}}{1-\varepsilon} \to 0$$ as $\varepsilon\to 0$ because $\alpha<1$.

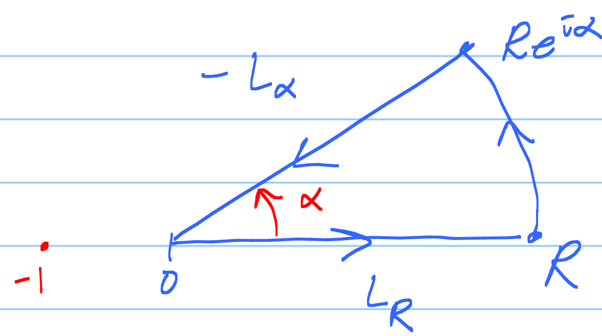
$$\underbrace{\int_{L_T} + \int_{L_B}}_{\to\,(1-e^{-i2\pi\alpha})\,I} + \underbrace{\int_{-C_\varepsilon}}_{\to\,0} + \underbrace{\int_{C_R}}_{\to\,0} = 2\pi i\, e^{-i\alpha\pi}$$

Let $\varepsilon\to 0$.
Then let $R\to\infty$.

So $$I = 2\pi i\,\frac{e^{-i\alpha\pi}}{1-e^{-i\alpha 2\pi}}\cdot\frac{e^{i\alpha\pi}}{e^{i\alpha\pi}} = \pi\left[\frac{2i}{e^{i\alpha\pi}-e^{-i\alpha\pi}}\right]$$

$$= \frac{\pi}{\sin\alpha\pi}$$

<u>Next</u>: $$\int_0^\infty \frac{1}{x^3+1}\,dx$$



L_α: $z(t) = t e^{i\alpha}$ $\quad 0\le t\le R$. $$\int_{-L_\alpha} dz = -\int_0^R \frac{1}{(e^{i\alpha}t)^3+1}\, e^{i\alpha}\,dt$$

Aha! Want $e^{i3\alpha}=1$,