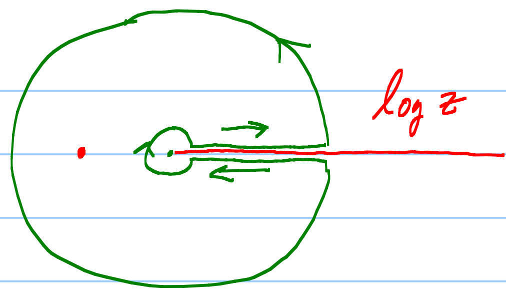
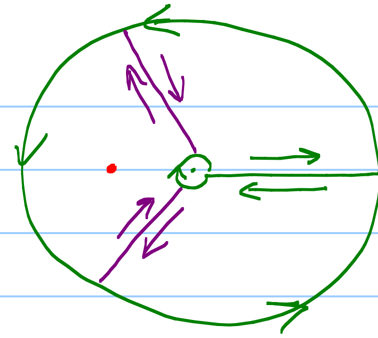
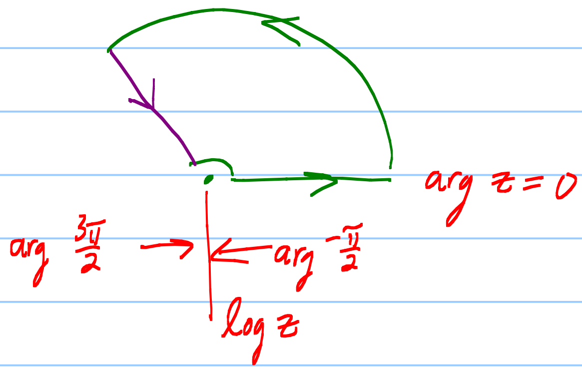
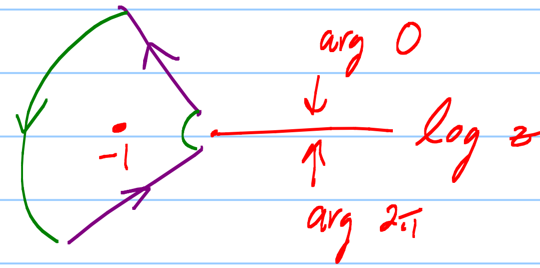
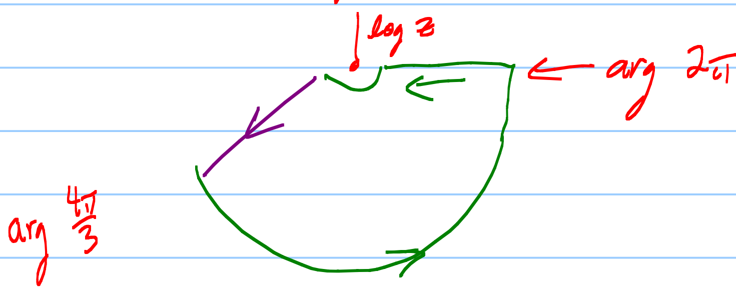
Lecture 22 Schwarz reflection



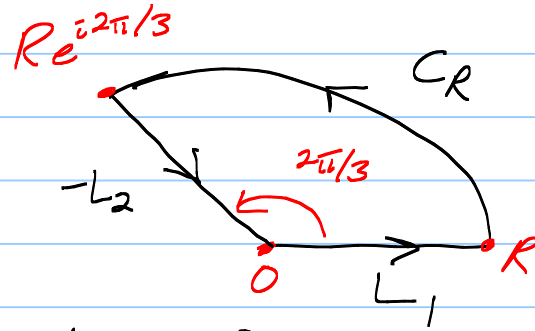








Purple parts cancel.



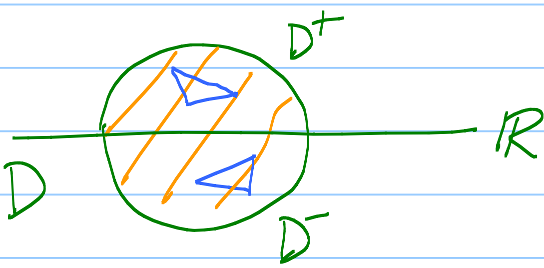
Practice problem: $\int_0^\infty \frac{1}{x^3+1}\,dx$

Want $\int_{L_1} f(z)\,dz$. Show $\int_{C_R} f(z)\,dz \to 0$

L_2: $z(t) = te^{i2\pi/3}$

$\int_{-L_2} f(z)\,dz = -\int_{L_2} f(z)\,dz$ = const times int. want

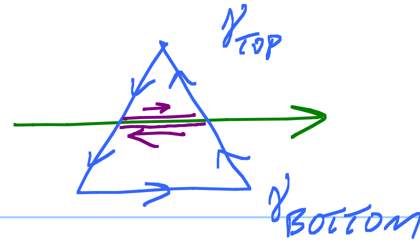
Lemma:



f continuous on D, analytic on $D^+ = \{z \in D : \operatorname{Im} z > 0\}$ and analytic on D^-.

Then f analytic on D.

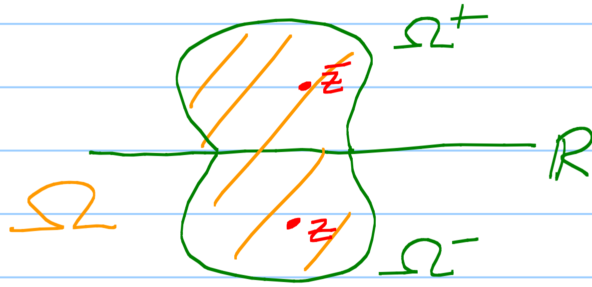
Key: Morera's with Δ's.



$$\int_{\Delta} = \int_{\gamma_{TOP}} + \int_{\gamma_{BOTTOM}}$$

$$\int_{\gamma_{TOP}} f\,dz = \lim_{\varepsilon \to 0} \underbrace{\int_{\gamma_{TOP}} f(z+i\varepsilon)\,dz}_{=0 \text{ by Cauchy's Thm on convex.}} = 0$$

<u>Schwarz Reflection Principle:</u>



Ω symmetric about $\mathbb{R}$.
f continuous on Ω^+ up to $\mathbb{R}$ and $f(z) \in \mathbb{R}$ when $z \in \mathbb{R}$. Furthermore, f is analytic on Ω^+.

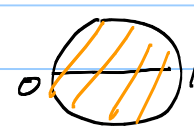
Then $F(z) = \begin{cases} f(z) & z \in \Omega^+ \text{ and } z \in \mathbb{R}. \\ \overline{f(\bar{z})} & z \in \Omega^- \end{cases}$ is analytic on Ω.

<u>Pf</u>: HWK Prob $\Rightarrow$ $\overline{f(\bar{z})}$ is analytic on Ω^-.

F is continuous on $\mathbb{R}$ from above <u>and</u> <u>below</u> because

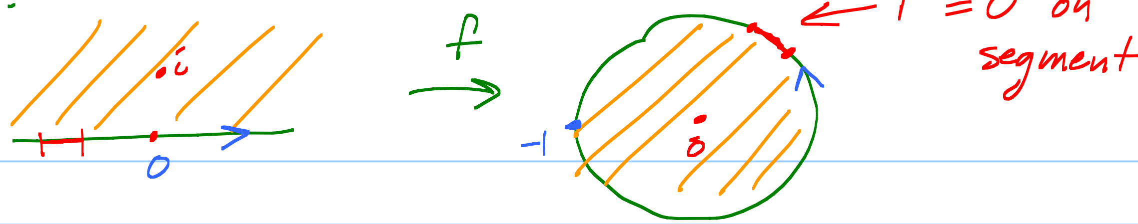
$\overline{f(\bar{x})} = \overline{f(x)} = f(x)$ when $x \in \mathbb{R}$. Lemma $\Rightarrow$ F analytic on Ω.

<u>Application</u>: Suppose f is analytic on the UHP and continuous up to $(0,1) \subset \mathbb{R}$. If f vanishes on $(0,1)$, then $f \equiv 0$ on the UHP.

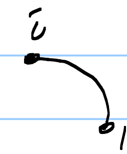
Why: Reflection fcn F is analytic on  and vanishes on $(0,1)$, a set with limit pts in the disc. Identity Thm $\Rightarrow F \equiv 0$. So $f \equiv 0$ on . $\Rightarrow f \equiv 0$ on UHP.

Important tool:

$$\frac{z-i}{z+i} = f(z)$$



$$F(f(z)) \equiv 0 \implies F \equiv 0 \text{ on disc.}$$



Question: Assume f cont on $\overline{D_1(0)}$ and $f \equiv 0$ on and analytic on $D_1(0)$. Show $f \equiv 0$.

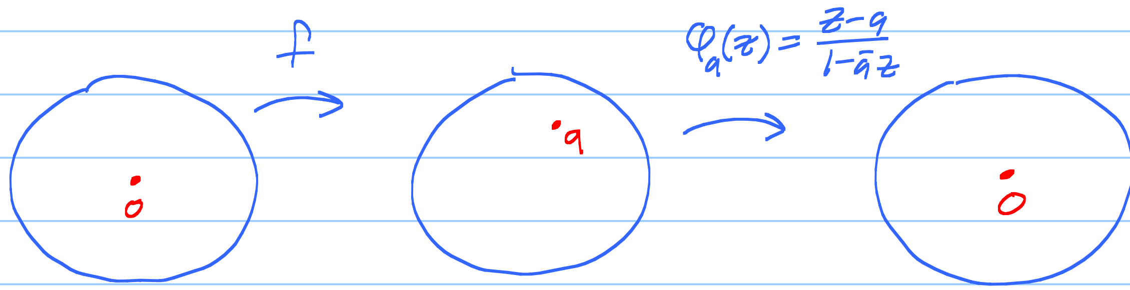
Trick: $F(z) = f(z)f(iz)f(-z)f(-iz)$ analytic on $D_1(0)$, cont on $\overline{D_1(0)}$ and $\equiv 0$ on bndry. Max Princ $\implies F \equiv 0$.

Pick $z_0 \in D_1(0)$ and take $z_n \to z_0$, z_n distinct.

For each n, one of four vanishes at z_n. One of the four must vanish at ∞ many z_n's. Limit pt of zeroes $\implies$ that one is $\equiv 0 \implies f \equiv 0$.

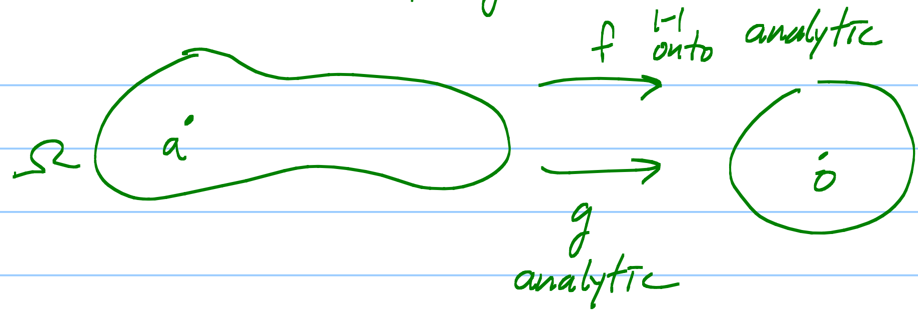
Cor of Schwarz Lemma: If $f : D_1(0) \to D_1(0)$ and is analytic, then $|f'(0)| \le 1$. Furthermore, if $|f'(0)| = 1$, then $f(z) = \lambda z$ where λ is a unimodular constant.

Pf:



Aha! Schwarz:
$$\left| \frac{d}{dz}(\varphi_a \circ f) \right|_{z=0} \le 1$$

Key to Riemann Mapping Theorem:



$f(a)=0$

$\boxed{g(a)=0}$ (*)

Schwarz $\Rightarrow$ $|g'(a)| \le |f'(a)|$. HWK Prob.

Cor to Schwarz allows us to drop (*):

$f : \Omega \to D_1(0)$ analytic, 1-1, onto and $f(a)=0$

$g : \Omega \to D_1(0)$ analytic

then $|g'(a)| \le |f'(a)|$. (Equality $\Rightarrow g = \lambda f$)