Lecture 23 Review (Exam in class, Monday, March 6)

Prac Prob 3: Know f, g analytic on $D_\varepsilon(a)$ and f has a simple zero at $z=a$, then $\mathrm{Res}_a \frac{g}{f} = \frac{g(a)}{f'(a)}$.
If f has a double zero, what?

$f(a)=0,\ f'(a)=0,$ but $f''(a)\neq 0.$

$$f(z) = \underset{a_0=0}{a_0} + \underset{a_1=0}{a_1}(z-a) + \underset{a_2=\frac{f''(a)}{2!}}{a_2}(z-a)^2 + \underset{a_3=\frac{f'''(a)}{3!}}{a_3}(z-a)^3 + \cdots$$

$$= (z-a)^2\Big[\underbrace{a_2 + a_3(z-a) + \cdots}_{F(z)\text{ analytic on } D_\varepsilon(a).}\Big]$$

Now
$$\frac{g(z)}{f(z)} = \frac{1}{(z-a)^2}\Big[\underbrace{\frac{g(z)}{F(z)}}_{=A_0+A_1(z-a)+\cdots}\Big] \leftarrow F(a)=a_2=\frac{f''(a)}{2}\neq 0$$

$$= \frac{A_0}{(z-a)^2} + \frac{A_1}{(z-a)} + \underbrace{A_2 + A_3(z-a)+\cdots}_{\text{analytic on } D_\varepsilon(a)}$$

Residue!

Aha!
$$\mathrm{Res}_a \frac{g}{f} = A_1 = \frac{\frac{d}{dz}\left[\frac{g(z)}{F(z)}\right]\Big|_{z=a}}{1!} = \frac{g'(a)F(a) - g(a)F'(a)}{F(a)^2}$$

Note: From above $F(a) = a_2 = \frac{f''(a)}{2}$ and $F'(a) = a_3 = \frac{f'''(a)}{6}$.

EX: $\mathrm{Res}_0 \frac{e^z}{\sin^2 z}$

Prob: $f: D_1(0) \circlearrowleft$ analytic, $f(0)=0$ with zero of order N, Show that $|f(z)| \le |z|^N$. Equality?

$$f(z) = a_N z^N + \cdots = z^N \underbrace{[a_N + a_{N+1} z + \cdots]}_{}$$

$F(z)$ analytic on $D_1(0)$

$F(0) = a_N = \frac{f^{(N)}(0)}{N!}$

$$F(z) = \begin{cases} \frac{f(z)}{z^N} & z \neq 0 \\ a_N & z = 0 \end{cases}$$

Pick $z_0 \in D_1(0)$.

Take radius r, $|z_0| < r < 1$.

Max Princ: $\max\limits_{|z|\le r} |F| = \max\limits_{|z|=r} \frac{|f(z)|}{|z|^N} < \frac{1}{r^N} \searrow 1$ as $r \nearrow 1$.

So $|F(z_0)| \le 1$. Get $|f(z)| \le |z|^N$ since z_0 arb.

And $|a_N| \le 1$. So $|f^{(N)}(0)| \le N!$.

If $|f(z)| = |z|^N$ for some $z \neq 0$ or if

$|f^{(N)}(0)| = N!$, then Max Princ $\Rightarrow F \equiv \lambda$. $|\lambda| = 1$.

or: Schwarz for f, $\frac{f}{z}$, $\frac{f}{z^2}$, ..., $\frac{f}{z^{N-1}}$ Done!

Prac Prob 9. Suppose $|f(z)| \le C(1 + |z|^N)$ for all z. Show that f is a poly of deg N.

Key: Cauchy Estimates $|f^{(n)}(z_0)| \le \frac{n!\,M}{r^n}$

where $M = \max\limits_{|z-z_0|=r} |f|$

$$\left| f^{(n)}(0) \right| \le \frac{n!\,(\text{Max } |f| \text{ on } C_r(0))}{r^n}$$

$$\le \frac{n!\, C(1+r^N)}{r^n} \to 0 \text{ as } r \to \infty$$
$$\text{if } n > N.$$

So $f^{n}(0) = 0$ for $n > N$. Power series for f poly of deg N or less.

2009: #4. $c|z|^N \le |f(z)|$ if $|z| > R$.

Hint: $\frac{1}{f} \to 0$ as $|z| \to \infty$. Look at Partial Fractions pf.

#3. $|f(z)| \le \frac{1}{\sqrt{|z|}}$ 0 isolated sing of f.

So $\underbrace{|z f(z)| \le \sqrt{|z|}}_{(*)}$ ← $z f(z)$ bounded near 0.

RRST ⇒ 0 is removable.

$z f(z) = H(z)$ for $z \ne 0$ where H analytic on $D_\varepsilon(0)$.

$(*) \Rightarrow \lim_{z \to 0} H(z) = 0$. So $H(0) = 0$. So $H(z) = z^N h(z)$
$h(0) \ne 0$.

$z f(z) = z^N h(z)$ for $z \ne 0$.

$f(z) = \underbrace{z^{N-1} h(z)}_{\text{analytic on } D_\varepsilon(0)}$ for $z \ne 0$. So 0 is removable for f.

Prob: g has an essential sing at 0 and f has an isolated sing at 0 and $f \not\equiv 0$.

Assume $|f(z)| \le |g(z)|$ near 0.

Show that f has an ess. sing at 0.

Key: RRST: $\left|\frac{f(z)}{g(z)}\right| \le 1$ $(*)$

Step 1: $g \not\equiv 0 \Rightarrow$ zeroes of g are Isolated.

If $g(z_0)=0$, $(*)$ shows $\frac{f}{g}$ bounded near z_0.

RRST $\Rightarrow z_0$ removable for $\frac{f}{g}$.

So $\frac{f}{g} = H$, where H is analytic on $D_\varepsilon(0) - \{0\}$.

Step 2: $(*) \Rightarrow H$ has a removable sing at 0.

Step 3: $\frac{f}{g} = H$ where $g(z) \neq 0$, $z \neq 0$.

$g = \frac{f}{H}$

Aha! If f has a removable sing or a pole at $z=0$, then so does g.

Cases: $H(0) \neq 0$. Easy

$H(z)=0$. $H(z) = z^N h(z)$, $h(0) \neq 0$.

f has a pole: $f(z) = z^{-M} F(z)$, $F(0) \neq 0$.