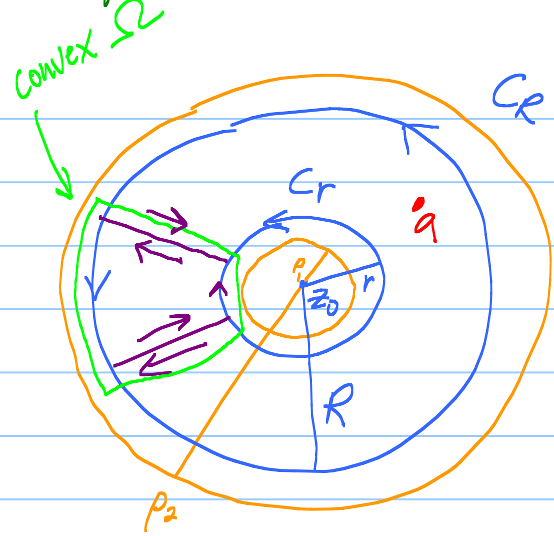
Lecture 26 Laurent expansions

Suppose f analytic on

$A(\rho_1, \rho_2) = \{z : \rho_1 < |z - z_0| < \rho_2\}$



Cauchy Thm for an annulus: $\left(\int_{C_R} - \int_{C_r}\right) f \, dz = 0$

Pf: Use Cauchy on convex, Add up pieces of pie.

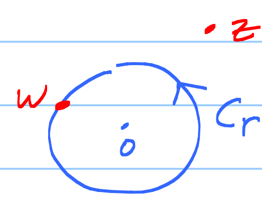
Cauchy Integral Formula: $\frac{1}{2\pi i}\left(\int_{C_R} - \int_{C_r}\right) \frac{f(z)}{z-a}\, dz = f(a)$

Pf: Apply Cauchy Thm to $F(z) = DQ = \frac{f(z) - f(a)}{z-a}$ ← r. sing at a

Laurent series: Take $z_0 = 0$.

$$f(z) = \underbrace{\frac{1}{2\pi i}\int_{C_R} \frac{f(w)}{w-z}\, dw}_{=\text{ power series } \sum_{n=0}^{\infty} a_n (z-z_0)^n} - \underbrace{\frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{w-z}\, dw}_{I(z)}$$

$$a_n = \frac{1}{2\pi i}\int_{C_R} \frac{f(w)}{(w-z_0)^{n+1}}\, dw$$



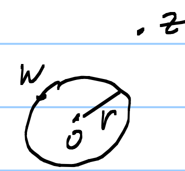
$\left|\frac{w}{z}\right| < 1$

$$I(z) = -\frac{1}{2\pi i}\int_{C_r} f(w)\left(-\frac{1}{z}\right) \cdot \frac{1}{1 - \frac{w}{z}}\, dw$$

Note $= \dfrac{1}{1-\tau} = \displaystyle\sum_{n=0}^{N} \tau^n + \underbrace{\frac{\tau^{N+1}}{1-\tau}}_{E_N(\tau)}$

$$I(z) = \frac{1}{2\pi i}\cdot\frac{1}{z}\int_{C_r} f(w)\left[\sum_{n=0}^{N}\left(\frac{w}{z}\right)^n + E_N\!\left(\frac{w}{z}\right)\right] dw$$

$$= \sum_{n=1}^{N+1} \underbrace{\left(\frac{1}{2\pi i}\int_{C_r} f(w)\, w^{n-1}\, dw\right)}_{a_{-n} = \frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{w^{-n+1}}\, dw \ \text{(Same as power series formula!)}} \frac{1}{z^n} + \mathcal{E}_N(z)$$



Finally, $\displaystyle |\mathcal{E}_N(z)| = \left|\frac{1}{2\pi i}\,\frac{1}{z}\int_{C_r} f(w) E_N\!\left(\frac{w}{z}\right) dw\right|$

$$\leq \frac{1}{2\pi}\,\frac{1}{|z|}\left(\max_{C_r}|f|\right)\frac{\left(\frac{r}{|z|}\right)^{N+1}}{1-\frac{r}{|z|}}\cdot 2\pi r$$

$\to 0$ uniformly on $\{z : |z-z_0| > \rho > r\}$

<u>Theorem:</u> f analytic on $D_\rho(z_0) - \{z_0\}$, then

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\substack{\text{unif conv on} \\ |z-z_0| < R < \rho \\ \text{analytic on } D_\rho(z_0)}} + \underbrace{\sum_{n=1}^{\infty} a_{-n}\frac{1}{(z-z_0)^n}}_{\substack{\text{unif conv on} \\ |z-z_0| > r > 0 \\ \text{analytic on } \mathbb{C} - \{z_0\}.}}$$

Write: $\displaystyle\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ converges means $\displaystyle\sum_{0}^{\infty}$ and $\displaystyle\sum_{-1}^{-\infty}$ both converge.

Def^n: Isolated sing at a: $\operatorname{Res}_a f = a_{-1}$

Theorem: f has an isolated sing at a.

1) $a_{-n} = 0$ for $n = 1, 2, 3, \ldots \iff a$ is removable.
2) Only finitely many negative terms $a_{-n} \neq 0 \iff a$ pole
3) Infinitely many negative terms $a_{-n} \neq 0 \iff a$ essential.

Pf: (1) $(\Rightarrow)$: $f(z)$ = power series. ✓

$(\Leftarrow)$ $a_{-n} = \frac{1}{2\pi i} \int_{C_r} f(w)\,(w-a)^{n-1}\,dw = 0$ by Cauchy's

removable sing at a

(2): Fact: Laurent expansions are <u>unique</u>.

<u>Why</u>: $a_N = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)}{(z-a)^{N+1}}\,dz$ ← same for any ρ $r < \rho < R$. (By C. Thm.)

$$= \frac{1}{2\pi i} \int_{C_\rho} \underbrace{\left[\sum_{n=-\infty}^{\infty} b_n (z-a)^n \right]}_{\text{conv unif on } C_\rho} \frac{1}{(z-a)^{N+1}}\,dz$$

$$= \sum_{n=-\infty}^{\infty} b_n \underbrace{\left(\frac{1}{2\pi i} \int_{C_\rho} (z-a)^{n-N-1}\,dz \right)}_{\text{all } = 0 \text{ except one, } n=N.} = b_N \checkmark$$

(2): $(\Rightarrow)$ easy.

$(\Leftarrow)$ Principal part gives the Laurent expansion.

(3): Left over case. ✓

EX: $\sin\frac{1}{z} = \frac{1}{z} - \frac{1}{3!}\frac{1}{z^3} + \frac{1}{5!}\frac{1}{z^5} - \cdots$

is the L. exp. See $z=0$ is an ess. sing.

Note: $\mathrm{Res}_a f = a_{-1} = \frac{1}{2\pi i}\int_{C_\rho} f(w)\,dw$

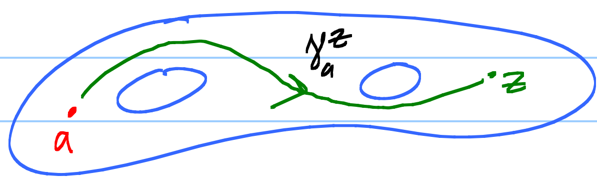
Meaning of Residue at iso. sing a. f analytic on $D_\rho(a) - \{a\}$.

Then $\mathrm{Res}_a f = 0 \iff f = F'$ on $D_\rho(a) - \{a\}$.

Lemma: $g = F'$ on a domain $\Omega \iff \int_\gamma g\,dz = 0$ for every closed curve in Ω.

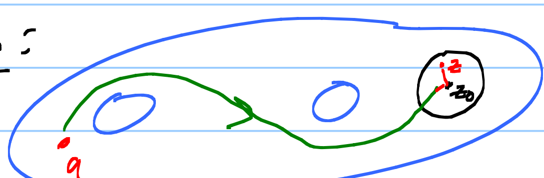
Pf: ($\Rightarrow$) $\int_\gamma F'\,dz = F(\text{END}) - F(\text{START}) = 0$

($\Leftarrow$) Try to define $F(z) = \int_{\gamma_a^z} g(w)\,dw$



Well defined: $\gamma_a^z \cup (-\tilde{\gamma}_a^z)$ closed curve.

Hyp $\Rightarrow$ $\int_{\gamma_a^z} = \int_{\tilde{\gamma}_a^z}$ ✓

Step 2: 

$\gamma_a^z = \gamma_a^{z_0} \cup L_{z_0}^z$

line

Aha! $\frac{d}{dz}\left(\int_{\gamma_q^{z_0}} + \int_{L_{z_0}^{z}}\right) g\,dw = 0 + g(w)$ ✓

<u>Pf of Theorem</u>: ($\Rightarrow$) $f = F'$. $\quad a_{-1} = \frac{1}{2\pi i}\int_{C_\rho} F'\,dw = 0$

($\Leftarrow$). Assume $\mathrm{Res}_a f = 0$. Let γ be closed in $D_\rho(a) - \{a\}$.

$$\int_\gamma f\,dz = \int_\gamma \sum_{n=-\infty}^{\infty} a_n z^n\,dz = \sum \int = 0$$

unif conv on γ $\qquad$ because all $\int = 0$ for $n \neq -1$.

$a_{-1} = 0$ takes care of the $n=-1$ <u>one</u>.