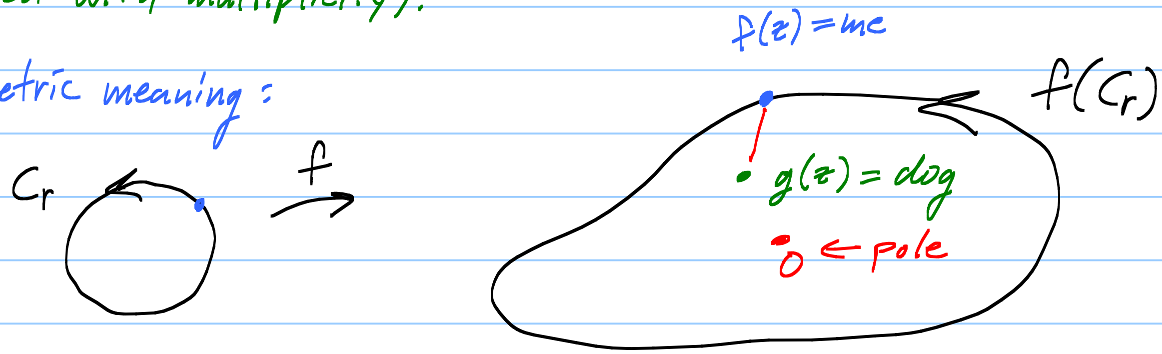
Lecture 27 Rouché's Theorem, Homotopy

Rouché's Theorem for a disc (or Toy region): Suppose f and g analytic on $D_R(a)$ and $0 < r < R$. If $|f(z) - g(z)| < |f(z)|$ on $C_r(a)$, then f and g have the same # of zeroes inside $C_r(a)$ (counted with multiplicity).

Geometric meaning:



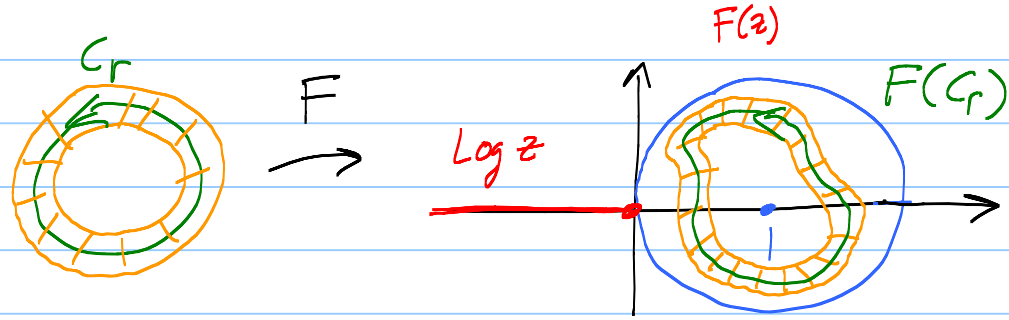
$$\underbrace{|f(z) - g(z)|}_{\text{leash length}} < \underbrace{|f(z)|}_{\text{dist to pole}}$$

Arg Princ: # zeroes of f in C_r
= # times $f(z)$ spins around origin
= # times dog goes around!

Pf: (*) $|f(z) - g(z)| < \underbrace{|f(z)|}_{\text{never zero on } C_r}$ on C_r

$g(z)$ never zero on C_r either: $|f(z) - 0| < |f(z)|$ ↯

Divide (*) by $|f(z)|$: $\quad \left|1 - \underbrace{\frac{g(z)}{f(z)}}_{F(z)}\right| < 1$ on C_r



$$\int_{C_r} \frac{F'}{F}\, dz = \int_{C_r} \frac{d}{dz}(\text{Log } F)\, dz = 0$$

But $\dfrac{F'}{F} = \dfrac{(g/f)'}{(g/f)} = \dfrac{g'}{g} - \dfrac{f'}{f}$

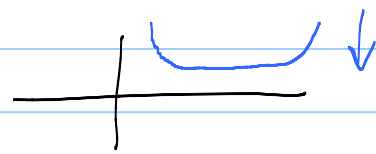
Aha! So $\displaystyle\int_{C_r} \frac{g'}{g}\,dz = \int_{C_r} \frac{f'}{f}\,dz.$ Argument Princ!

Remark: Can allow finitely many poles inside: get (# zeroes) − (# poles) same.

Theorem: (Hurwicz Thm) Suppose f_n is a seq of analytic fcns on a domain Ω. If the f_n are non-vanishing on Ω, and $f_n \rightrightarrows f$ on Ω. Then either

1) f is non-vanishing (analytic), or
2) $f \equiv 0$ on Ω.

Remark: Way false $\mathbb{R} \to \mathbb{R}$.



EX: $\Omega = D_1(0) - \{0\}$, $f_n(z) = z^n$ $\quad f_n \rightrightarrows f \equiv 0$.

Pf: Suppose $f(z_0) = 0$, $z_0 \in \Omega$. If $f \not\equiv 0$, z_0 is an isolated zero of f. So $\exists$ $\overline{D_R(z_0)} \subset \Omega$ such that z_0 is the only zero of f in $\overline{D_R(z_0)}$. Let $m = \min\limits_{C_R(z_0)} |f|$. $(m > 0)$.

$$\underbrace{|f_n(z) - f(z)| < m}_{\text{if } n > N_0} \leq |f(z)| \text{ on } C_R \text{ if } n > N_0,$$

Rouché's $\Longrightarrow$ $\underset{\substack{\uparrow \\ \text{no zeroes}}}{f_n}$ and $\underset{\substack{\uparrow \\ \text{at least one zero}}}{f}$ have same # zeroes in C_R.

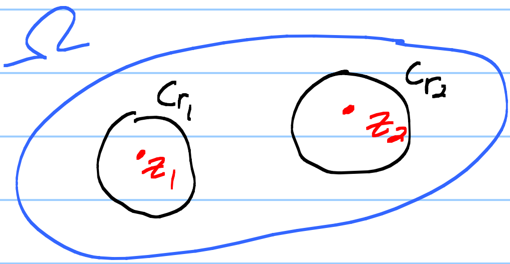
$\lightning$. Done.

Corollary: (Hurwicz #2): $f_n \rightrightarrows f$. If each f_n is one-to-one, then either

A) f is one-to-one, or

B) $f \equiv$ const on Ω.

Pf: Redo pf for Hurwicz #1: Suppose $f(z_1) = f(z_2) = w_0$

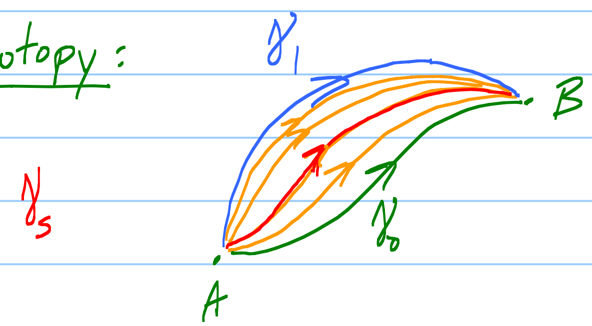


$F(z) = f(z) - w_0$ ← zeroes at z_1 and z_2

Rouché's $\Rightarrow$ $f_n - w_0$ ← has zeroes in each disc for $n > \mathrm{Max}(N_1, N_2)$.

So f_n not 1-1. ↯.

Homotopy:



$\gamma_j : z_j(t), \quad 0 \le t \le 1$

$\gamma_0 \sim \gamma_1$ (γ_0 is homotopic to γ_1)

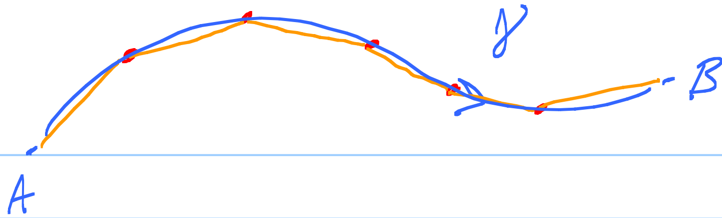
means there is a continuous fcn $H: [0,1] \times [0,1] \to \Omega$

such that
$$H(0,t) = z_0(t), \quad 0 \le t \le 1$$
$$H(1,t) = z_1(t), \quad 0 \le t \le 1$$

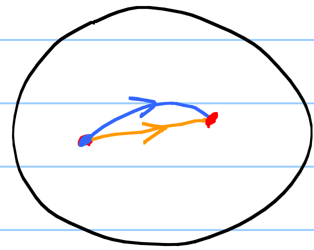
and
$$H(s,0) = A \quad \text{for } 0 \le s \le 1$$
$$H(s,1) = B \quad \text{for } 0 \le s \le 1.$$

Think $\gamma_s : z_s(t) = H(s,t)$.

Easy fact: Given a continuous curve γ, can approx it by a piecewise linear curve:



Hmmm.



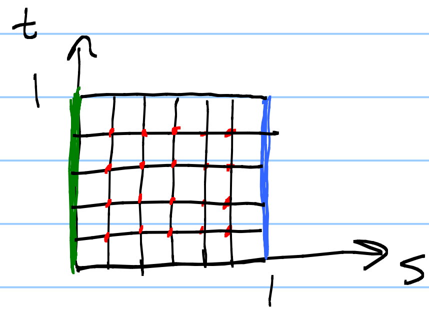
Cauchy on convex

$f = F'$

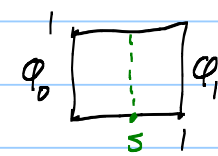
$\int_{\text{blue}} f \, dz = \int_{\text{orange}} f \, dz$

Assume γ_0 and γ_1 are piecewise C^1-smooth.

Fact: Can approx a cont homotopy by piecewise C^1-homotopy.



Step 1: Linearize in t direction

Step 2: 

reparametrize to get 1×1 square.

$$\varphi_s(t) = (1-s)\varphi_0(t) + s\varphi_1(t)$$

Get piecewise linear homotopy H.