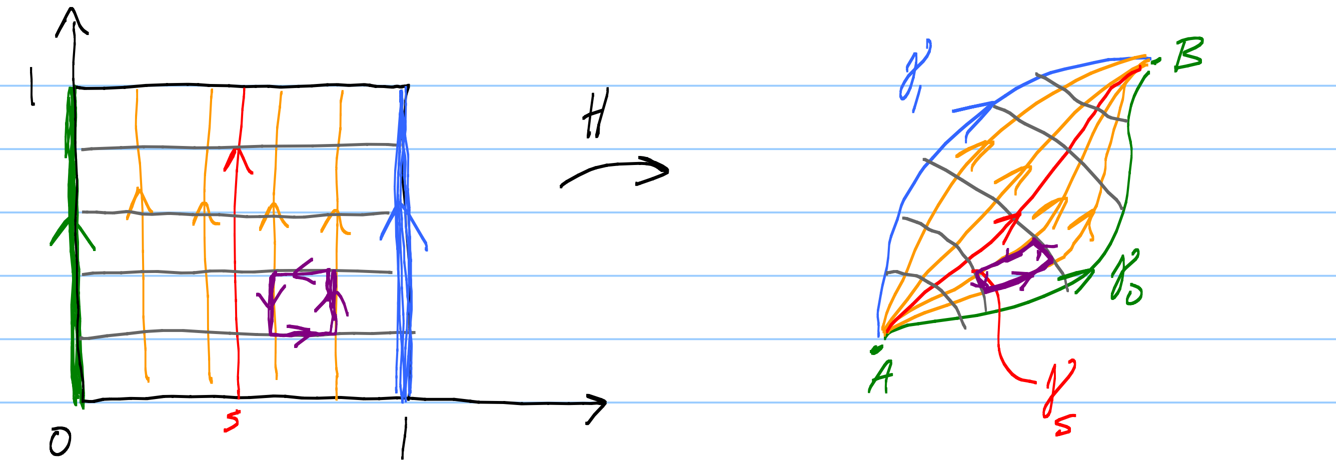
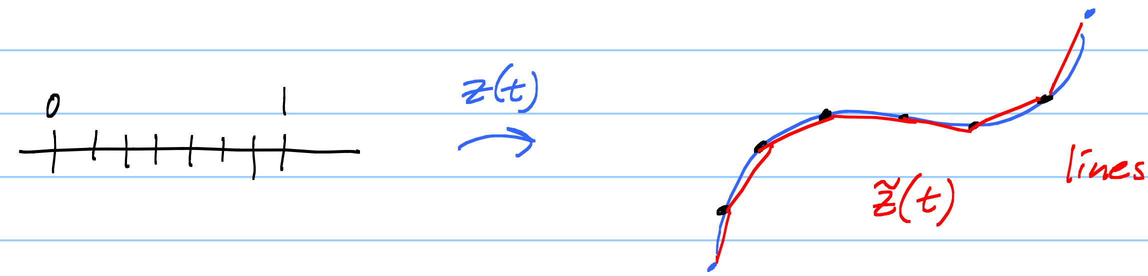
# Lecture 28 Homotopy, Simply connected domains, the Cauchy Thm

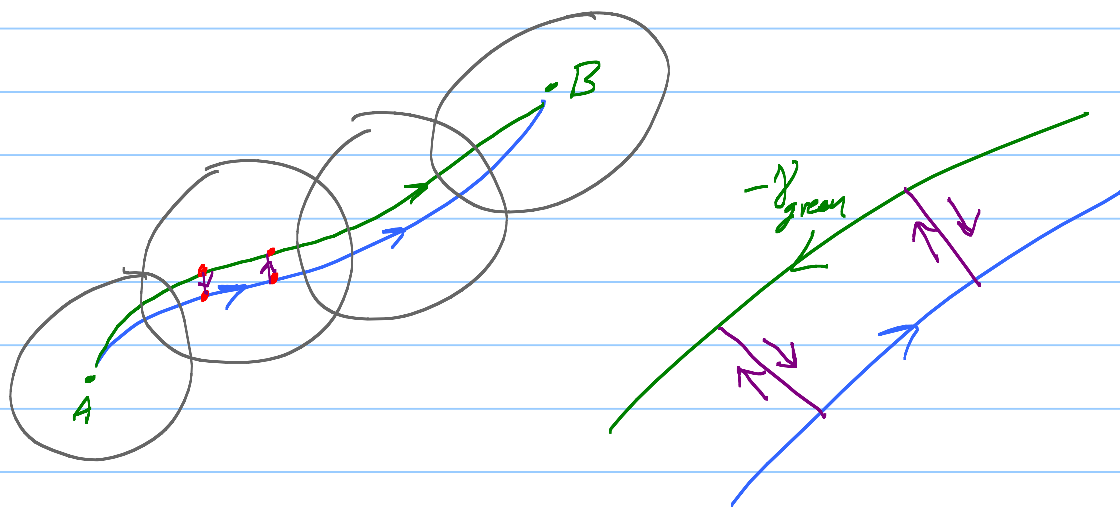



EX:




$$\tilde{z}(t) = z(t_n) + \frac{t - t_n}{t_{n+1} - t_n}\left[z(t_{n+1}) - z(t_n)\right]$$

Stein:




Big Theorem: Suppose $f$ is analytic on a domain $\Omega$ and $\gamma_0 \sim \gamma_1$ (starting at $A$ and ending at $B$), then

$$\int_{\gamma_0} f\,dz = \int_{\gamma_1} f\,dz.$$

**Def$^n$:** Two <u>closed</u> curves $\gamma_0$ and $\gamma_1$ in $\Omega$ are homotopic in $\Omega$ (write $\gamma_0 \sim \gamma_1$ or $\gamma_0 \sim_\Omega \gamma_1$) if there is a homotopy $H: [0,1] \to [0,1] \to \Omega$ such that

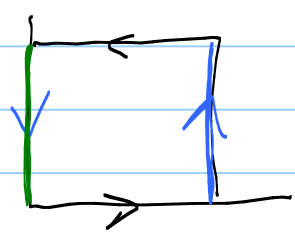
$$\begin{cases} H(0,t) = z_0(t) \leftarrow \gamma_0 \\ H(1,t) = z_1(t) \leftarrow \gamma_1 \end{cases}$$

$$H(s,0) = H(s,1) \leftarrow \gamma_s : \ z_s(t) = H(s,t)$$

are <u>closed</u>.

<u>Big Theorem</u>: $\int_{\gamma_0} f\, dz = \int_{\gamma_1} f\, dz$ if $\gamma_0 \sim \gamma_1$ are <u>closed</u> homotopic in $\Omega$.

<u>Pf</u>:



Top cancels the bottom!

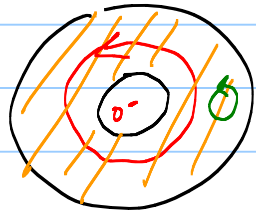
**Def$^n$:** A domain $\Omega$ is <u>simply connected</u> if every closed curve in $\Omega$ is closed homotopic "to a point", meaning a constant curve $z(t) \equiv z_0 \in \Omega$.

<u>The Cauchy Theorem</u>: If $f$ is analytic on a simply connected domain, then $\int_\gamma f\, dz = 0$ for any closed

curve $\gamma$ in $\Omega$.

Pf: $\gamma \sim z_0$ $\qquad \int_\gamma f\, dz = \int_0^1 f(z_0) \cdot 0\, dt = 0$

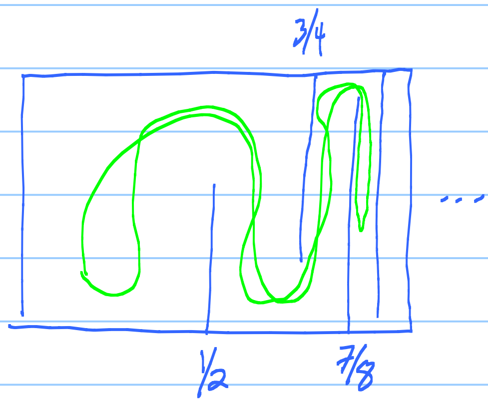
EX:

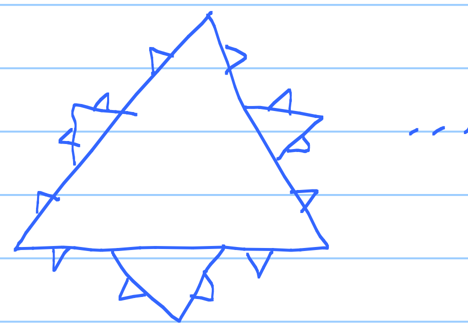


Not simply connected.

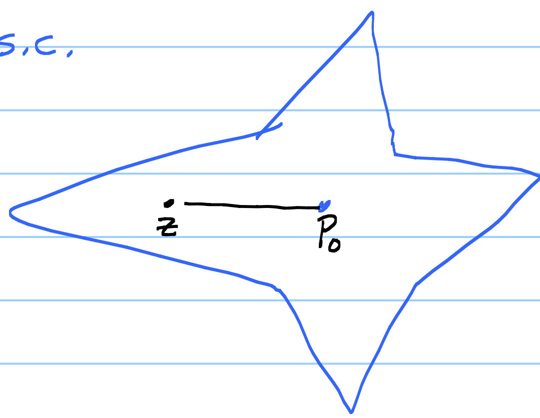
$$\int_\gamma \frac{1}{z}\, dz = 2\pi i$$

EX:

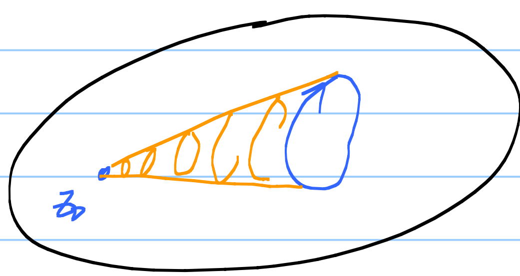



Star shaped regions are s.c.




Convex domains are s.c.




Thms: Suppose $f$ analytic on a s.c. domain $\Omega$.

A) Then $f$ has an analytic antiderivative $F$ with $F' = f$.

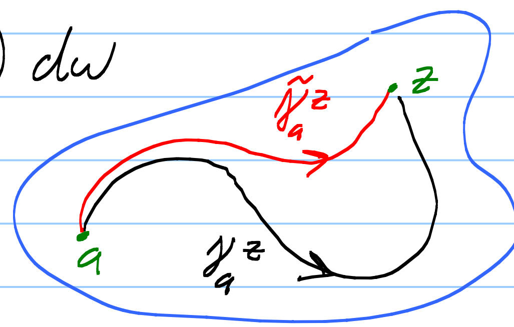
B) Furthermore, if $f$ is <u>non-vanishing</u> on $\Omega$, then $\exists$ analytic $G$ on $\Omega$ with $f = e^G$. Also

$$f = \left(e^{G/N}\right)^N$$

$\leftarrow$ $H$ is an $N$-th root of $f$ on $\Omega$.

<u>Pf</u>: A)

$$F(z) = \int_{\gamma_a^z} f(w)\, dw$$

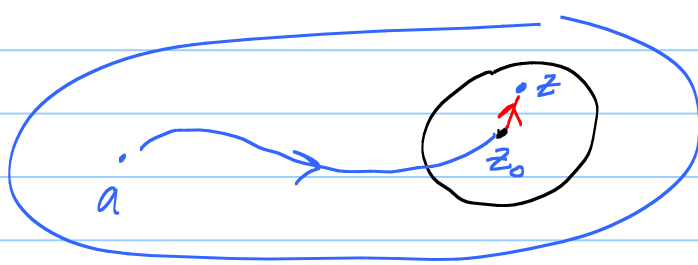



$\Gamma = \gamma_a^z \cup \left(-\tilde{\gamma}_a^z\right)$ <u>closed</u>.

$\Omega$ s.c. $\Rightarrow \int_\Gamma f\, dz$. So $\int_{\gamma_a^z} f\, dw = \int_{\tilde{\gamma}_a^z} f\, dw$

So $F$ <u>well</u> defined,

To show $F' = f$, restrict attention to disc $D_r(z_0) \subset \Omega$.




Line $L_{z_0}^z$

Take $\gamma_a^z = \gamma_a^{z_0} \cup L_{z_0}^z$

$$F(z) = \int_{\gamma_a^z} + \int_{L_{z_0}^z}$$

$$F'(z) = 0 + f(z)$$ $\leftarrow$ done on convex, Goursat.

B) $\displaystyle g(z) = \int_{\gamma_a^z} \frac{f'(w)}{f(w)}\, dw$, $\quad G = g + c$

**Theorem:** If $u$ is harmonic on a s.c. domain $\Omega$, then $u$ has a harm conj on $\Omega$.

**Pf:** $u$ harmonic $\Rightarrow$ $f = u_x - i u_y$ analytic on $\Omega$

[ Locally, on a disc $D_r(z_0) \subset \Omega$, get $g = u + iv$.

$$g' = u_x + i v_x \underset{\substack{CR \\ Eqns}}{=} u_x - i u_y$$

Let $F$ be an anti-der for $f = u_x - i u_y$.

**Claim:** $v = \operatorname{Im} F$ is it.

Write $F = U + i V$

$$F' = U_x + i V_x = U_x - i U_y = u_x - i u_y.$$

So $\nabla U = \nabla u$ on $\Omega$. So $U = u + c$ on $\Omega$.

Aha! $u + iV$ is analytic.

Review Arzela-Ascoli Theorem: Rudin p. 158.

Ahlfors: p. 219-227. Stein p. 225