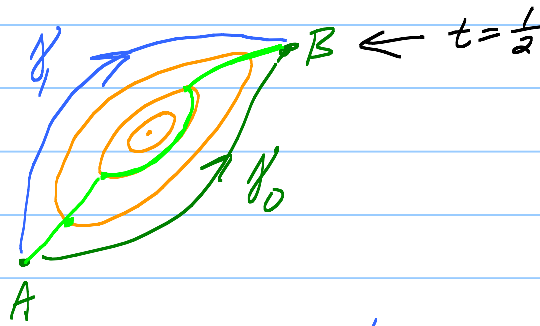
Lecture 29 Montel's Theorem

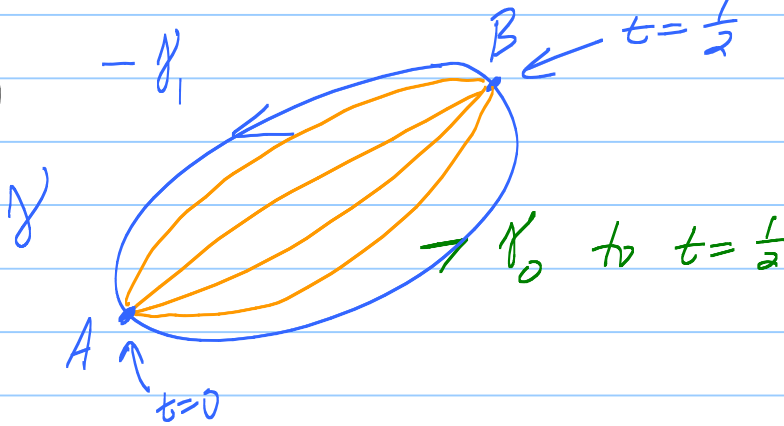
<u>Fact</u>: Ω is a simply connected domain $\Longleftrightarrow$ any two curves that start at same pt and end at same pt are homotopic.

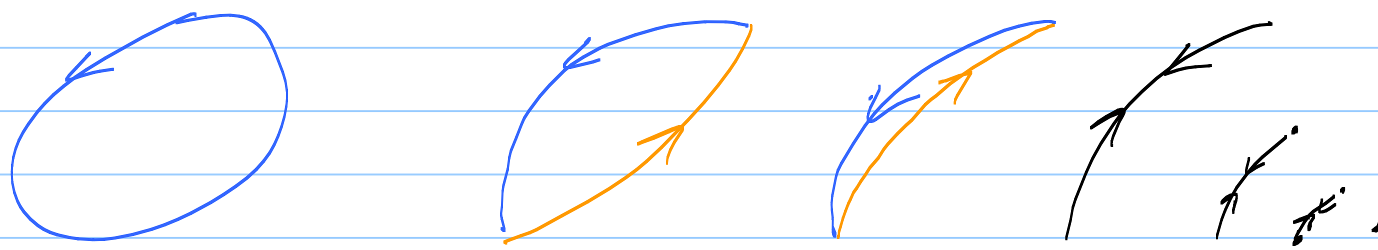
$(\Longrightarrow)$

$\Gamma = \gamma_0 \cup (-\gamma_1)$
closed



$(\Longleftarrow)$

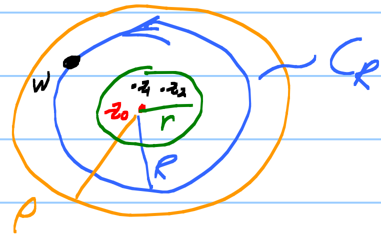




<u>Cor</u>: $\int_{\gamma_A^B} f\, dz$ only depends on A and B on a simply conn. domain (f analytic).

<u>Lemma</u>: Uniformly bounded families of analytic fcns on a domain are uniformly equicontinuous on compact subsets.

<u>Why</u>:



f analytic on $D_\rho(z_0)$

$0 < r < R < \rho$

$z_1, z_2 \in D_r(z_0)$

$$\left| f(z_1) - f(z_2) \right| = \left| \frac{1}{2\pi i} \int_{C_R} f(w) \left[\frac{1}{w - z_1} - \frac{1}{w - z_2} \right] dw \right|$$

$$= \frac{1}{2\pi} |z_1 - z_2| \left| \int_{C_R} f(w) \frac{1}{(w - z_1)(w - z_2)} \, dw \right|$$

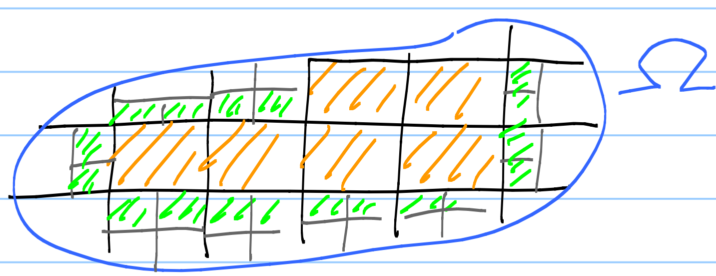
$$\leq \frac{1}{2\pi} |z_1 - z_2| \underbrace{\left(\max_{C_R} |f| \right)}_{M} \frac{1}{(R - r)^2} \cdot (2\pi R)$$

$f \in \mathcal{F} \leftarrow$ family all bnd by M. Given $\varepsilon > 0$, can find a $\delta > 0$ such that $|f(z_1) - f(z_2)| < \varepsilon$ if $|z_1 - z_2| < \delta$ with $z_1, z_2 \in D_r(z_0)$.

Cover compact set by finitely many $D_{r_j}(z_j)$ like the above.

<u>Montel's Theorem</u>: Suppose f_n are analytic fcns on a domain Ω that are uniformly bounded on compact subsets of Ω. Then there is a subseq that converges unif on compact subsets to an analytic fcn f.

<u>Pf</u>; <u>Step 1</u>: Ω has an "exhaustion by compacts"



K_1 = union of closed squares in $\Omega \cap D_{10}(0)$

K_2 = same, half side length $\cap D_{20}(0)$

K_3 = same, half side length again $\cap\ D_{40}(0)$

$\vdots$

Get K_j compact, $K_1 \subset K_2 \subset K_3 \cdots$, $\bigcup K_j = \Omega$.

<u>Step 2</u>: Fix compact $K = K_j$. Let $\{z_n\}$ be a countable dense subset of K (rational Re and Im parts, e.g.)

<u>Step 3</u>: $f_1(z_1), f_2(z_1), f_3(z_1), \ldots$ unif bnd.

So $\exists$ conv subseq:

$$f_{1,1}(z_1),\ f_{1,2}(z_1), \ldots.$$

Now $f_{1,1}(z_2), f_{1,2}(z_2), \ldots$ is bdd.

$\exists$ conv subseq

$$f_{2,1}(z_2),\ f_{2,2}(z_2), \ldots.$$

Continue. Aha! $f_{n,n}$ is a subseq that converges at each z_j!

<u>Step 4</u>: Let $F_n = f_{n,n}$. Claim: F_n is uniformly Cauchy on K, and therefore converges unif on K.

$$|F_n(z) - F_m(z)| = \Big|\underbrace{F_n(z) - F_n(z_j)}_{\text{close by equicont.}} + \underbrace{F_n(z_j) - F_m(z_j)}_{F_n(z_j) \text{ converges. So Cauchy}} + \underbrace{F_m(z_j) - F_m(z)}_{\text{close}}\Big|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$

Cover K by $D_\delta(w_k)$, $w_k \in K$, take a finite subcover.

Pick <u>one</u> z_j in each $D_\delta(w)$. Middle $\frac{\varepsilon}{3}$ get for $n, m > \operatorname{Max} N_k$

Finally, get $F_{1,1}, F_{1,2}, \ldots$ conv on K_1

Take subseq $F_{2,1}, F_{2,2}, \ldots$ conv on K_2

$\ldots,$

$F_{n,n}$ converges uniformly on each K_j.

Note: If $K \subset \Omega$ is compact, then $\exists\, j$ such $K \subset K_j$.

Riemann Mapping Thm plan: $\Omega \neq \mathbb{C}$ s.c. domain. Pick $a \in \Omega$,

Assume Ω bounded. $\tilde{\mathcal{F}}$ = analytic fcns $f: \Omega \to D_1(0)$ one-to-one.

Take a seq $f_n \in \tilde{\mathcal{F}}$ so that $|f_n'(a)| \to \sup_{f \in \tilde{\mathcal{F}}} |f'(a)|$ ← finite by Cauchy ests.

Montel ⟹ $\exists$ conv sub seq $f_{n_k} \rightrightarrows F$

Hurwicz ⟹ F is 1-1.

Last step: Schwarz Lemma will be key in seeing that F is onto.

Square roots on s.c. domains needed too.