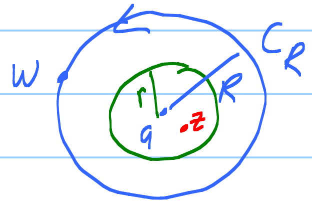
Lecture 30 Riemann Mapping Theorem

Thm: Ω simply connected domain $\neq \mathbb{C}$. There is an analytic fcn $f: \Omega \to D_1(0)$ that is one-to-one and onto. [Ω is conformally equivalent to $D_1(0)$!]

Thm: If f_n are analytic on a domain Ω and $f_n \rightrightarrows f$ on Ω, then $f_n^{(k)} \rightrightarrows f^{(k)}$ too!

Way false $\mathbb{R} \to \mathbb{R}$. Limit could be nowhere diff'ble.

Pf:



$$\left|f_n^{(k)}(z) - f^{(k)}(z)\right| = \left|\frac{k!}{2\pi i}\int_{C_R} \frac{f_n(w)-f(w)}{(w-z)^{k+1}}\,dw\right|$$

$$\leq \frac{k!}{2\pi}\Big(\underset{\substack{C_R \\ \uparrow \text{compact}}}{\mathrm{Max}} |f_n - f|\Big) \cdot \frac{1}{(R-r)^{k+1}} \cdot (2\pi R) \longrightarrow 0 \text{ unif on } D_r(a).$$

Given compact K, cover by finitely many $D_{r_i}(a_i)$.

Pf of R.M.T.: Let $\mathcal{F}$ be the family of analytic fcns h

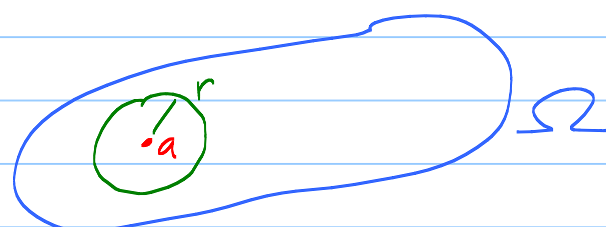
$h: \Omega \to D_1(0)$ that are one-to-one.

Claim: $\mathcal{F}$ is not empty. [Pf. Later. Easy if Ω bndd. Then $\frac{z}{R}$ is in $\mathcal{F}$ for R big.]

Pick $a \in \Omega$. Let $M = \mathrm{Sup}\{|h'(a)| : h \in \mathcal{F}\}$.

Step 1: $0 < M < \infty$

$\uparrow$ h is 1-1, so $h'(a) \neq 0$



Cauchy Estimate: $\quad |h'(a)| \leq \dfrac{1!\, \mathrm{Max}\{|h| : C_r(a)\}}{r^1} \leq \dfrac{1}{r}$

[Can let $r \nearrow \mathrm{dist}(a, b\Omega)$. $\quad |h'(a)| \leq \dfrac{1}{\mathrm{dist}(a, b\Omega)}$.]

Step 2: Take a seq $h_n \in \tilde{\mathcal{F}}$ such that $|h_n'(a)| \to M$.

All h_n bndd by 1. Montel's $\Rightarrow$ can assume

$h_n \rightrightarrows f$ on Ω. [f is R. map!]

Step 3: Today's Thm $\Rightarrow$ $h_n'(a) \to f'(a)$. So $|f'(a)| = M$.

Note: $M \neq 0$. So f is not constant.

Hurwicz #2 $\Rightarrow$ f is one-to-one.

Step 4: $f \in \tilde{\mathcal{F}}$. Need to show $f : \Omega \to D_1(0)$.

$|h_n(z)| < 1$. So $|f(z)| \le 1$. If $|f(z_0)| = 1$ for some

$z_0 \in \Omega$, then Max Modulus Thm $\Rightarrow$ $|f|$ has a max at z_0

$\Rightarrow$ $f \equiv$ const. ↯.

Step 5: f is onto. Suppose not. Suppose w_0 missed.

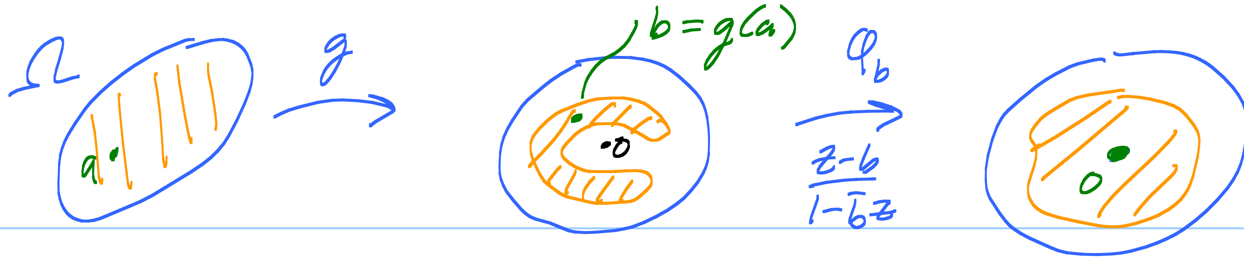
$\varphi_{w_0} \circ f$ is a non-vanishing analytic on s.c. Ω. There is an analytic square root fcn g on Ω such that $g^2 = \varphi_{w_0} \circ f$.

Claim: $g \in \tilde{\mathcal{F}}$.

Fact: g^2 1-1 $\Rightarrow$ g 1-1. ✓ $g(z_1) = g(z_2)$

$g(z_1)^2 = g(z_2)^2 \Rightarrow z_1 = z_2$

$|g^2(z)| < 1 \Rightarrow |g(z)| < 1$. ✓



<u>Claim</u>: $\varphi_b \circ g$ has a strictly bigger derivative than f at a!

So w_0 does not exist, $\lightning$

Note $\varphi_b \circ g \in \tilde{\mathcal{F}}$. Write $\tilde{f} = \varphi_b \circ g$.

Check that $f = F \circ \tilde{f}$ where

$$F = \varphi_{w_0}^{-1} \circ s \circ \varphi_b^{-1} \quad \text{where} \quad s(z) = z^2.$$

<u>Note</u>: $F : D_1(0) \to D_1(0)$, but F is <u>not</u> 1-1 because of the z^2. Corollary to Schwarz $\Rightarrow |F'(0)| < 1$

Finally: $f'(a) = F'(\underbrace{\tilde{f}(a)}_{0})\, \tilde{f}'(a)$

$$M = |f'(a)| = \underbrace{|F'(0)|}_{<1}\, |\tilde{f}'(a)| < |\tilde{f}'(a)|$$

bigger than M. $\lightning$

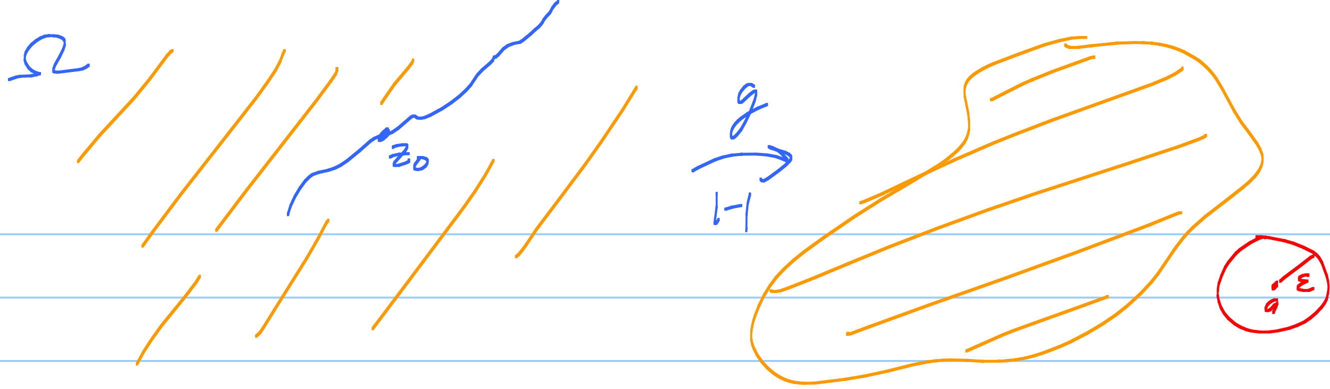
So f is onto. Done!

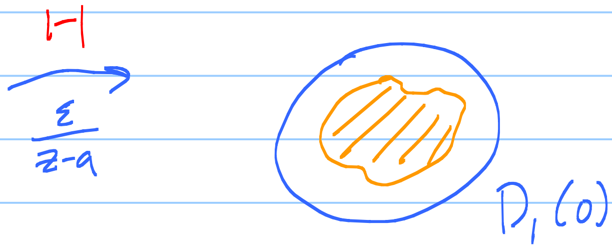
<u>Last piece</u>: $\tilde{\mathcal{F}}$ not empty.

$\Omega \neq \mathbb{C}$. So $\exists\, z_0 \in \mathbb{C} - \Omega$. So $z - z_0$ is non-vanishing on s.c. Ω. So $\exists$ analytic square root fcn g on Ω with $g(z)^2 = z - z_0$. $z - z_0$ 1-1 $\Rightarrow g$ 1-1.

<u>Claim</u>: $g(\Omega)$ misses a disc.

Assume for now.

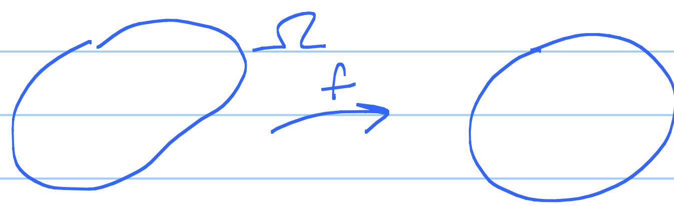




See that $\frac{\varepsilon}{g(z)-a}$ is in $\tilde{\mathcal{F}}$,

Why g misses a disc. Open M. Thm $\Rightarrow$ $g(\Omega)$ contains a disc $D_\delta(b)$. Show that $-D_\delta(b)$ is not in $g(\Omega)$ because $g^2 = z - z_0$ is 1-1.

Other facts:



Ω has C^∞ smooth bndry, then $f \in C^\infty(\bar{\Omega})$.

If bndry of Ω is a Jordan curve, then f is continuous up the bndry and maps bndry 1-1 onto the unit circle. [Carathéodory's Thm.]